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Strategy-proof allocation problem with hard budget constraints and income effects: weak efficiency and fairness

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Abstract: We consider the single-object allocation problem with monetary transfers. Agents have hard budgets and their utility functions may exhibit income effects. We characterize truncated Vickrey rules with endogenous reserve prices by constrained efficiency or weak envy-freeness for equals, in addition to individual rationality, no subsidy for losers, and strategy-proofness. The same characterization result hold even if we replace weak envy freeness for equals with other fairness conditions; equal treatment of equals, envy-freeness, and anonymity in welfare.

JEL classification: D47, D63, D82

Keywords: Single-object allocation problem, Non-quasi-linear preference, Hard budget constraint, Efficiency, Fairness, Strategy-proofness, Vickrey rule with reserve prices

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Strategy-proof allocation problem with hard budget constraints and income effects: weak efficiency and fairness*

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Abstract

We consider the single-object allocation problem with monetary transfers. Agents have hard budgets and their utility functions may exhibit income effects. When hard budget constraints exist, it is known that *efficiency* and *strategy-proofness* are incompatible along with *individual rationality* and *no subsidy*. Our objectives are (i) to explore which forms of partial *efficiency* are compatible with *strategy-proofness*, and (ii) to investigate *strategy-proof* rules that satisfy some *fairness* conditions, instead of *efficiency*. We focus on *constrained efficiency* as a weak efficiency condition, and on *weak envy-freeness for equals* as a fairness condition. We introduce *truncated Vickrey rules with endogenous reserve prices*. Some of them may fail to satisfy either *constrained efficiency*, *weak envy-freeness for equal*, or *strategy-proofness*. In order to satisfy the properties, we impose additional conditions: *exclusive tie-breaking rules*, *non-negligible reserve prices*, and *prioritized tie-breaking rules*, respectively. First we show that the truncated Vickrey rule with endogenous reserve prices and with an exclusive and prioritized tie-breaking rule are the only rules satisfying *constrained efficiency*, *individual rationality*, *no subsidy for losers*, and *strategy-proofness*. Next we show that the truncated Vickrey rules with non-negligible endogenous reserve prices and with a prioritized tie-breaking rule are the only rules satisfying *weak envy-freeness for equals*, *individual rationality*, *no subsidy for losers* and *strategy-proofness*. This characterization also holds when replacing *weak envy-freeness for equals* with *equal treatment of equals* or *envy-freeness*. Moreover, if endogenous reserve prices are upper anonymous, then the same result holds for *anonymity in welfare*.

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1 Introduction

In real-life auctions, participants often face budget constraints, which is the maximum amount of money they can spend on auctioned objects. An important example of budget-constrained auctions is a spectrum license auction (Bulow et al. 2009), where firms must put aside money in advance to acquire spectrum licenses. The presence of budget constraints significantly complicates firms' bidding strategies and challenges the application of existing theoretical frameworks that do not consider budget constraints.

It is one of crucial goals for auctioneers to allocate objects *efficiently* to agents. Moreover, to accurately evaluate *efficiency* based on true preferences, auctioneers require *strategy-proofness*, where reporting true utility functions becomes a dominant strategy. If utility functions are quasi-linear, and so there is no budget constraint, then the Groves rules are the only rules satisfying *efficiency* and *strategy-proofness* (Holmström, 1979). However, Dobzinski et al. (2012) demonstrate that if private budgets exist, which undermines quasi-linearity of utility functions, then no rule satisfies these conditions along with *individual rationality* and *no subsidy*.

Building upon the seminal work of Dobzinski et al. (2012), subsequent literature has investigated budget-constrained auctions, striving to achieve positive results. There are various approaches to reconcile the conflict between *efficiency* and *strategy-proofness*. One approach is to restrict the domain of preferences. Dobzinski et al. (2012) show that if budgets are public information, then there is a rule satisfying *efficiency*, *individual rationality*, *no subsidy*, and *strategy-proofness*. Mackenzie and Zhou (2022) also establish a similar positive result when budgets are private information, but payments are discrete and agents demand at most one object (unit-demand).

Alternatively, some studies opt to compromise either *efficiency* or *strategy-proofness*. Le (2018) relaxes these conditions and constructs a rule that satisfies *efficiency* and *strategy-proofness* for "almost all preferences" in addition to *individual rationality* and *no subsidy*.

We must note that the studies mentioned above do not account for the income effects experienced by agents. In large-scale auctions such as spectrum license auctions, substantial payments can diminish agents' capacity to afford certain complementary goods related to the auctioned objects, thereby leading to significant income effects. Thus, the presence of income effects, alongside budget constraints, constitutes a crucial aspect of real-life auctions. However, there is limited literature on non-quasi-linear utility due to the complexities introduced by income effects in the analysis.

In this paper, we consider an environment where both budget constraints and income effects are present, aiming to derive positive results by relinquishing *efficiency*. One natural approach to achieving positive results is to moderate *efficiency*. While full *efficiency* is incompatible with strategy-proofness, certain degrees of partial *efficiency* may be reconcilable. Our primary objective is to explore which forms of partial *efficiency* are compatible with *strategy-proofness*, while ensuring *individual rationality* and *no subsidy*.

As an alternative approach, we impose fairness conditions, instead of *efficiency*, such as *envy-freeness*, *anonymity in welfare*, *equal treatment of equals*, and others. Although society often prioritizes these fairness conditions, there is limited literature examining budget-constrained auctions that simultaneously satisfy fairness conditions and maintain *strategy-proofness*. Our secondary objective is to investigate whether there is a rule that satisfies certain fairness conditions and *strategy-proofness*, in addition to *individual rationality* and *no subsidy*.

1.1 Result

We consider the single-object allocation problem with monetary transfers. Each agent has a utility function which depends on the assignment of the object and the payment. Our formulation is so general that agents' utility functions can exhibit *income effect*. Additionally, each agent has a *hard budget constraint*, wherein payments exceeding the budget render their utility as negative infinity. We call the set of utility functions the *domain*. A *utility profile* is a vector consisting agents' utility functions. An *allocation* specifies who gets the object and how much agents pay.

The willingness to pay is called the *valuation*. We define the minimum between the willingness to pay and the budget as the *truncated valuation*. When receiving the object and paying the budget is strictly better than receiving no object with no payment for an agent, we say that the *budget constraint is binding*. This is because in this case, the truncated valuation is equal to the budget.

A *rule* is a mapping from a set of utility profiles to the set of allocations. A rule satisfies *individual rationality* if each agent's outcome is at least as good as receiving no object and paying nothing. A rule satisfies *no subsidy for losers* if each agent who receives no object makes the nonnegative payment. We focus on *weak envy-freeness for equals* as a fairness condition, introduced by Sakai (2013a). It requires that if two agents with identical utility functions differ in outcome (one being a winner, the other a loser), then the loser does not prefer the winner's outcome. This condition is so weak that it is implied by many other fairness conditions, such as *equal treatment of equals*, *envy-freeness*, *anonymity in welfare*, and so on.

When budget constraints exist, Vickrey rules may violate *individual rationality* because some agent may pay the money exceeding his/her budget. To address this issue, Le (2018) proposed *truncated Vickrey rules*, which realize the outcome of a Vickery rule where inputs are truncated valuations, not valuations. He demonstrates that this rule satisfies *efficiency* and *strategy-proofness* for almost all utility profiles. However, when these properties need to be guaranteed for all profiles, Le's formulation is insufficient. Therefore, we introduce a *truncated Vickrey rule with endogenous reserve prices* by extending the truncated Vickrey rule. In this extended version, each agent's reserve price is determined endogenously by other agents' utility functions.

In the presence of budget constraints, achieving both *efficiency* and *strategy-proofness*, in addition to *individual rationality* and *no subsidy for losers*, is impossible. However, we can aim for certain degrees of partial *efficiency*. To pursue this goal, we decompose efficiency into three conditions. The first condition is *within budget*, which mandates that each agent's payment does not exceed their budget. The second condition is *no wastage*, which ensures that the object is always assigned to an agent. The third condition is *constrained efficiency*, which demands that no reallocation can improve both agents' welfare and revenue simultaneously. In this study, we forgo *no wastage* and focus on developing a rule that satisfies *within budget* and *constrained efficiency*, in addition to *individual rationality*, *no subsidy for losers*, and *strategy-proofness*.

All truncated Vickrey rules with endogenous reserve prices satisfy *within budget*, *individual rationality*, and *no subsidy for losers*. However, some of them fail to satisfy either *constrained efficiency*, *weak envy-freeness for equals*, or *strategy-proofness*. Whether these conditions hold depends on the chosen tie-breaking rules and endogenous reserve prices.

Achieving *strategy-proofness* requires a tie-breaking rule to satisfy a *prioritized condition*. In this condition, if ties exist, the agent whose budget constraint is binding wins the object. Furthermore, *constrained efficiency* necessitates a tie-breaking rule satisfying an *exclusive condition*. The condition requires that when ties exist, the agent whose budget constraint is not biding loses the object.

Attaining *weak envy-freeness for equals* requires endogenous reserve prices to satisfy *non-negligibility condition*. The non-negligibility requires that for an agent, his/her reserve price is strictly larger than the maximum truncated valuation among other agents if the other agent with the highest truncated valuation has the binding budget constraint.

Our first results are about *efficiency*. We show that *the truncated Vickrey rules with endogenous reserve prices and with an exclusive and prioritized tie-breaking rule are the only rules satisfying constrained efficiency, individual rationality, no subsidy for losers, and strategy-proofness.* Furthermore, we show that such rules always violate *no wastage*, which prove the incompatibility between *efficiency* and *strategy-proofness* along with *individual rationality* and *no subsidy for losers*.

Our second results are about fairness. We show that *the truncated Vickrey rules with non-negligible endogenous reserve prices and with a prioritized tie-breaking rule are the only rules satisfying weak envy-freeness for equals, individual rationality, no subsidy for losers, and strategy-proofness.* We also show that the same result holds even when replacing *weak envy-freeness for equals* with *equal treatment of equals* or *envy-freeness*. Furthermore, if endogenous reserve prices are *upper anonymous*, then the same result holds for *anonymity in welfare*.

1.2 Related literature

Our contributions include (i) allowing agents to have hard budget constraints and (ii) considering *fair* and *strategy-proof* auctions. Previous studies often focus on one factor while neglecting the other, resulting in limited research that considers both. Furthermore, our study stands out due to its consideration of non-quasi-linear utility functions, which has been often overlooked in existing research. In the following, we explain the existing literature about each factor, respectively.

Budget-constrained auctions: On the no income effect domain with homogenous objects, Dobzinski et al. (2012) show that when utility functions exhibit constant marginal valuation and budgets are private information, then there is no rule satisfying *efficiency*, *individual rationality*, *no subsidy*, and *strategy-proofness*. However, they also establish that when budgets are public information, the modified version of Ausubel's (2004) clinching auction satisfies these conditions, and moreover it is unique if there are two agents. However, on the domain violating constant marginal valuation, the impossibility result returns even if budgets are public information (Lavi and May 2012; Ting and Xing 2012; Yi 2023).

There are some extensions of the result of Dobziski et al. (2012). Fiat et al. (2011) consider single-valued preferences, which include preferences exhibiting constant marginal valuation as a special case. In this situation, they show that when budgets are public information, single-valued combinatorial auctions satisfy *efficiency*, *individual rationality*, *no subsidy*, and *strategy-proofness*. Hirai and Sato (2023) study about polymatroidal environments, where the feasible allocations of objects are restricted by some monotone and submodular function. In this situation, they show that if budgets are public information, then polyhedral clinching auctions satisfy *efficiency*, *individual rationality*, *no subsidy*, and *strategy-proofness*.

Mackenzie and Zhou (2022) investigate the situation where budgets are private information but each agent demands at most one object (unit-demand). On the no income effect domain with homogenous objects, they show that if payments are discrete, then efficient pendulum auctions satisfy *efficiency*, *individual rationality*, *no subsidy*, and *obvious strategy-proofness*.

On the no income effect domain with heterogenous objects, Dütting, Henzinger, and Starnberger (2015) establish the negative result. They show that even when budgets are public information, there is no rule satisfying *efficiency*, *individual rationality*, *no subsidy*, and *strategy-proofness*. In contrast with their result, Aggarwal et al. (2009) establish the positive result by limiting preferences to those with unit-demand. On the no income effect and unit demand domain with heterogenous objects, they show that when budgets are private information and utility profiles are in general position,[1] the stable matching rule satisfies these four conditions. Dütting, Henzinger, and Weber (2015) also show the positive result by extending the result of Aggarwal et al. (2009).

Some studies give up *efficiency* or *strategy-proofness*, while keeping private budgets. On the greedy domain with heterogenous objects, Le (2017) shows that the iterative second price auctions are the only rules satisfying *non-wastefulness*, *symmetry*, *non-bossiness*, *individual rationality*, *no subsidy*, and *strategy-proofness*. He also shows that no rule

[1]This condition excludes the situation where two agents have the same budget simultaneously.

satisfies these conditions on the general domain. On the single-minded domain with heterogenous objects, Le (2018) shows that truncated VCG rules satisfy *efficiency* and *strategy-proofness* for almost all preferences in addition to *individual rationality* and *no subsidy*. Moreover, he shows that a rule satisfying these conditions is a truncated VCG rule almost everywhere. When objects are heterogenous and agents' preferences exhibit income effects, Shinozaki (2023) shows that modified pay as bid rules satisfy *efficiency*, *individual rationality*, *no subsidy*, and *non-obvious manipulability*.

Fair and strategy-proof auctions: When agents have quasi-linear preferences and there is a single object, Sakai (2013b) shows that the second price auction with a reserve price is the only rule satisfying *weak efficiency*, *non-imposition* and *strategy-proofness*. For the case of homogenous objects, Basu and Mukherjee (2022) characterize the same rule by *anonymity in welfare*, *non-imposition*, *strategy-proofness*, and some mild conditions.

On the general domain including the quasi-linear domain, Sakai (2013a) shows that when there is a single object, the Vickrey rule is characterized by *no wastage*, *weak envy-freeness for equals*, *non-imposition*, and *strategy-proofness*. Adachi (2014) shows the same result adding to *welfare continuity* for the case of homogenous objects.

On the general domain with a single object, Kazumura et al. (2017) shows that Vickrey rules with reserve prices are the only rule satisfying *anonymity in welfare*, *loser payment independence*, and *strategy-proofness*.

On the quasi-linear domain with heterogenous objects, Ashlagi and Serizawa (2012) show that the Vickrey rule is the only rule satisfying *no wastage*, *anonymity in welfare*, *individual rationality*, *no subsidy for losers* and *strategy-proofness*. On the same domain, Ohseto (2006) characterizes the rules that satisfy *no wastage*, *envy-freeness* and *strategy-proofness*.

Note that all of the studies mentioned here do not consider hard budget constraints.

1.3 Organization

The structure of this paper is as follows: Section 2 presents the fundamental components of the model. Following this, in Section 3, we introduce truncated Vickrey rules with endogenous reserve prices. Section 4 explains the results, comprising three parts concerning *strategy-proofness*, *weak efficiency*, and *fairness*. Lastly, Section 5 provides a conclusion. All proofs are relegated to Appendix.

2 The model

There are n agents and a single object. Let $N = \{1, 2, \cdots, n\}$ be the set of agents. We denote consuming the object and not consuming the object by 1 and 0, respectively. A typical (consumption) bundle for agent i is a pair $z_i = (x_i, t_i) \in \{0, 1\} \times \mathbb{R}$, where x_i is the consumption of the object and t_i is the payment for agent i.

Each agent has a utility function $u_i : \{0,1\} \times \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ such that (i) $u_i(0,0) = 0$ and (ii) there is a budget $b_i \in \mathbb{R}_+ \cup \{\infty\}$ such that for each $x_i \in \{0,1\}$ and each $t_i \in \mathbb{R}$, if $t_i \leq b_i$, then $u_i(x_i, t_i) \neq -\infty$, and otherwise, $u_i(x_i, t_i) = -\infty$. For the sake of convenience, if $t_i = \infty$, then we let $u_i(x_i, t_i) = -\infty$. We denote by $\mathcal{U}$ a typical class of utility functions, and call it a **domain**. We make the following assumptions.

1. *Finiteness:* If $u_i(1, b_i) \leq 0$, then there is $v_i \in \mathbb{R}_+$ such that $u_i(1, v_i) = 0$.

2. *Money monotonicity:* For each $x_i \in \{0,1\}$ and each $t_i, t'_i \in \mathbb{R}$, if $t_i < t'_i \leq b_i$, then $u_i(x_i, t_i) > u_i(x_i, t'_i)$.

3. *Object desirability:* For each $t_i \in \mathbb{R}$ with $t_i \leq b_i$, $u_i(1, t_i) > u_i(0, t_i)$.

Let $\mathcal{U}^C$ be the set of all utility functions satisfying the above three properties, and call it the **classical domain**. Given $u_i \in \mathcal{U}^C$, we call v_i defined by finiteness a **valuation** for u_i. Moreover, for the sake of notational convenience, if $u_i(1, b_i) > 0$, then let $v_i = \infty$. Note that by money monotonicity, the valuation v_i is uniquely determined. We call $\min\{v_i, b_i\}$ a **truncated valuation** for u_i. If $v_i = \infty$, we say that *agent i's budget constraint is binding*. Given $u_i \in \mathcal{U}^C$ and $t_i \in \mathbb{R}$, we define the **compensation** for u_i from t_i by $c_i(t_i) > 0$ such that $c_i(t_i) = t_i - b_i$ if $t_i > b_i$, and $u_i(0, t_i - c_i(t_i)) = u_i(1, t_i)$ otherwise. Throughout the paper, we consider a domain included by the classical domain, that is, $\mathcal{U} \subseteq \mathcal{U}^C$.

A **utility profile** is an n-tuple of agents' utility functions $u = (u_1, \ldots, u_n) \in \mathcal{U}^n$. Given $i \in N$ and $N' \subseteq N$, let $u_{-i} = (u_j)_{j \neq i}$ and $u_{-N'} = (u_j)_{j \in N \setminus N'}$. Given a utility profile $u \in \mathcal{U}^n$, let

$$N(u) = \left\{ i \in N : \min\{v_i, b_i\} \geq \max_{j \neq i} \min\{v_j, b_j\} \right\}$$

be the set of the agents who have the highest truncated valuation, and let

$$N^\infty(u) = \left\{ i \in N : \min\{v_i, b_i\} \geq \max_{j \neq i} \min\{v_j, b_j\} \text{ and } v_i = \infty \right\}.$$

be the set of the agents in $N(u)$ whose budget constraint is binding.

A feasible **object assignment** is an n-tuple $x = (x_1, \ldots, x_n)$ such that $\sum_{i \in N} x_i \leq 1$. Let X be the set of all feasible object assignments, that is, $X = \{(x_1, \ldots, x_n) \in \{0,1\}^n : \sum_{i \in N} x_i \leq 1\}$. An **allocation** is a pair of a feasible object assignment and a vector of payments, $z = ((x_1, x_2, \ldots, x_n), (t_1, t_2, \ldots, t_n)) \in X \times \mathbb{R}^n$. We denote the set of all allocations by $Z = X \times \mathbb{R}^n$. Given $z \in Z$ and $i \in N$, $z_i = (x_i, t_i)$ denotes the bundle of agent i.

A **rule** is a mapping $f = (x, t) \colon \mathcal{U}^n \to Z$. Given a rule f and a utility profile $u \in \mathcal{U}^n$, agent i's bundle under f at u is denoted by $f_i(u) = (x_i(u), t_i(u))$. Given $i \in N$ and $u_{-i} \in \mathcal{U}^{n-1}$, let $\mathcal{U}^W(u_{-i}) = \{u_i \in \mathcal{U} : x_i(u_i, u_{-i}) = 1\}$ be the set of i's utility functions for which he/she wins the object, and let $\mathcal{U}^L(u_{-i}) = \mathcal{U} \setminus \mathcal{U}^W(u_{-i})$.

Given $u \in \mathcal{U}^n$, an allocation $z \in Z$ is **efficient for** u if there is no $z' \in Z$ such that (i) for each $i \in N$, $u_i(z'_i) \geq u_i(z_i)$, (ii) $\sum_{i \in N} t'_i \geq \sum_{i \in N} t_i$, and (iii) at least one inequality in (i) and (ii) holds strictly. We say that z' **dominates** z **for** u if z' satisfies (i), (ii), and (iii) above. We similarly define *efficiency* as a condition imposed on a rule.

- **Efficiency**: For each $u \in \mathcal{U}^n$, $f(u)$ is efficient for u.

Next we define a weak fairness condition introduced by Sakai (2013a). It says that if two agents with identical utility functions differ in outcome (one being a winner, the other a loser), then the loser does not prefer the winner's outcome.

- **Weak envy-freeness for equals**: For each $u \in \mathcal{U}^n$ and each $i, j \in N$, if $u_i = u_j$, $x_i(u) = 0$ and $x_j(u) = 1$, then $u_i(f_i(u)) \geq u_i(f_j(u))$.

Weak envy-freeness for equals is so weak that it is implied by many fairness conditions. All of the conditions below are stronger than *weak envy-freeness for equals*.

- **Equal treatment of equals**: For each $u \in \mathcal{U}^n$ and each $i, j \in N$, if $u_i = u_j$, then $u_i(f_i(u)) = u_i(f_j(R))$.
- **Envy-freeness**: For each $u \in \mathcal{U}^n$ and each $i, j \in N$, $u_i(f_i(u)) \geq u_i(f_j(u))$.
- **Anonymity in welfare**: For each $u, u' \in \mathcal{U}^n$ and each $i, j \in N$, if $u_i = u'_j$, $u_j = u'_i$, and $u_{-\{i,j\}} = u'_{-\{i,j\}}$, then $u_i(f_i(u)) = u_i(f_j(u'))$.

Finally, we introduce basic properties.

- **Individual rationality**: For each $u \in \mathcal{U}^n$ and each $i \in N$, $u_i(f_i(u)) \geq 0$.
- **No subsidy for losers**: For each $u \in \mathcal{U}^n$ and each $i \in N$, if $x_i(u) = 0$, then $t_i(u) \geq 0$.
- **Strategy-proofness**: For each $u \in \mathcal{U}^n$, each $i \in N$ and each $u'_i \in \mathcal{U}$, $u_i(f_i(u)) \geq u_i(f_i(u'_i, u_{-i}))$.

3 Truncated Vickrey rules with endogenous reserve prices

In this section, we introduce *truncated Vickrey rules with endogenous reserve prices*. First we define a *Vickrey rule* (Vickrey 1961; Saitoh and Serizawa 2008; Sakai 2008).[2]

[2] In their definitions, the object is always assigned to some agent with the maximum valuation. However, in our definition, if ties exist, the object could not be assigned to any agent. In this sense, our definition does not precisely correspond to theirs.

Definition 1. A rule f on $\mathcal{U}^n$ is a **Vickrey rule** if for each $u \in \mathcal{U}^n$ and each $i \in N$,

$$x_i(u) = \begin{cases} 0 & \text{if } v_i < \max_{j \neq i} v_j \\ 1 & \text{if } v_i > \max_{j \neq i} v_j \end{cases},$$

and

$$t_i(u) = \begin{cases} 0 & \text{if } x_i(u) = 0 \\ \max_{j \neq i} v_j & \text{if } x_i(u) = 1 \end{cases}.$$

A Vickrey rule may violate *individual rationality*. To see this, let f be a Vickrey rule, and let $u \in \mathcal{U}^n$ and $i \in N$ be such that $v_i > \max_{j \neq i} v_j > b_i$. Then, by definition, $f_i(u) = (1, \max_{j \neq i} v_j)$. However, since $t_i(u)$ exceeds his/her budget b_i, $u_i(f_i(u)) = -\infty$, which is a violation of *individual rationality*. To overcome this problem, Le (2018) introduces a *truncated Vickrey rule*.

Definition 2. A rule f on $\mathcal{U}^n$ is a **truncated Vickrey rule** if for each $u \in \mathcal{U}^n$ and each $i \in N$,

$$x_i(u) = \begin{cases} 0 & \text{if } \min\{v_i, b_i\} < \max_{j \neq i} \min\{v_j, b_j\} \\ 1 & \text{if } \min\{v_i, b_i\} > \max_{j \neq i} \min\{v_j, b_j\} \end{cases},$$

and

$$t_i(u) = \begin{cases} 0 & \text{if } x_i(u) = 0 \\ \max_{j \neq i} \min\{v_j, b_j\} & \text{if } x_i(u) = 1 \end{cases}.$$

Any truncated Vickrey rule f satisfies *individual rationality* because for each $u \in \mathcal{U}^n$ and each $i \in N$ with $x_i(u) = 1$, $t_i(u) \leq \min\{v_i, b_i\}$. Note that if there is no budget constraint for each agent, that is, $b_i = \infty$ for each $i \in N$, then a truncated Vickrey rule coincides with a Vickrey rule.

Finally we introduce a *truncated Vickrey rule with endogenous reserve prices*, which is a generalization of a truncated Vickrey rule.

Given $i \in N$, i's **(reserve) price function** is $r_i : \mathcal{U}^{n-1} \to \mathbb{R} \cup \{\infty\}$, whose outputs are independent of i's utility functions. A **(reserve) price function profile** $r = (r_1, \ldots, r_n)$ is an n-tuple of price functions for each agent. Let $\mathcal{R}$ be a set of price function profiles.

Definition 3. Given a price function profile $r \in \mathcal{R}$, a rule f on $\mathcal{U}^n$ is a **truncated Vickrey rule with endogenous reserve prices** r if for each $u \in \mathcal{U}^n$ and each $i \in N$,

$$x_i(u) = \begin{cases} 0 & \text{if } \min\{v_i, b_i\} < \max\{\max_{j \neq i} \min\{v_j, b_j\}, r_i(u_{-i})\} \\ 1 & \text{if } \min\{v_i, b_i\} > \max\{\max_{j \neq i} \min\{v_j, b_j\}, r_i(u_{-i})\} \end{cases}, \tag{V-i}$$

and

$$t_i(u) = \begin{cases} 0 & \text{if } x_i(u) = 0 \\ \max\{\max_{j \neq i} \min\{v_j, b_j\}, r_i(u_{-i})\} & \text{if } x_i(u) = 1 \end{cases}. \tag{V-ii}$$

4 Results

This section consists of four subsections. In the first subsection, we investigate *strategy-proof* rules. In contrast to the case without budget constraints, some truncated Vickrey rules with endogenous reserve prices may violate *strategy-proofness*. Consequently, we establish a necessary and sufficient condition for these rules to satisfy *strategy-proofness*. Subsequently, we characterize the rules that satisfy *individual rationality*, *no subsidy for losers*, and *strategy-proofness*.

In the second subsection, we investigate *weak efficient* rules. We first show that *efficiency* is decomposed by three conditions: *within budget*, *no wastage*, and *constrained efficiency*. Next we show a necessary and sufficient condition for truncated Vickrey rules with endogenous reserve prices to satisfy *constrained efficiency*, and characterize these rules by *constrained efficiency*, *individual rationality*, *no subsidy*, and *strategy-proofness*. Finally we show that these rules always violate *no wastage*, which shows the incompatibility between *efficiency* and *strategy-proofness*.

In the third subsection, we investigate *fair* rules. We characterize the rules satisfying *weak envy-freeness for equals*, *individual rationality*, *no subsidy for losers*, and *strategy-proofness*. We also show that the same characterization result holds even if we replace *weak envy-freeness for equals* with the other fairness conditions; *equal treatment of equals*, *envy-freeness*, and *anonymity in welfare*.

In the last subsection, we investigate the independence of the properties.

Before going to these subsections, we provide some conditions on a domain. We first introduce a *unbounded truncated valuations* condition. It says that the set of truncated valuations of a domain has no upper bound. The class of these domains includes many economically meaningful domains; the no income effect domain, the positive income effect domain, and so on. They also include the cases where agents face no budget constraints (namely, $b_i = \infty$ for each $u_i \in \mathcal{U}$), but not include the cases with public budget constraints.

Definition 4. A domain $\mathcal{U}$ satisfies **unbounded truncated valutions** if for each $a \in \mathbb{R}_+$, there is $u_i \in \mathcal{U}$ such that $a < \min\{v_i, b_i\}$.

Next we introduce a *small compensation* condition. It says that for any two distinct payments, there exists a utility function such that the change of the compensation does not exceed the change of the payment.

Definition 5. A domain $\mathcal{U}$ satisfies **small compensation** if for each $t_i > 0$ and each $t'_i < t_i$, there is $u_i \in \mathcal{U}$ such that $0 < c_i(t'_i) - t'_i < t_i - t'_i$.

Thirdly we introduce a *density* condition. It says that for each non-negative two numbers, there exists a utility function whose truncated valuation is between the two numbers. Moreover, if the budget constraint is binding, we say that the *strong density* condition is satisfied.

Definition 6. A domain $\mathcal{U}$ satisfies **density** if for each $a, b \in \mathbb{R}_+$ with $a < b$, there is $u_i \in \mathcal{U}$ such that $a < \min\{v_i, b_i\} < b$. Furthermore, if there is $u_i \in \mathcal{U}$ such that $a < \min\{v_i, b_i\} < b$ and $v_i = \infty$, we say that $\mathcal{U}$ satisfies **strong density**.

Note that the above two conditions are stronger than the unbounded truncated valuations condition.[3] Also note that the strong density condition excludes the quasi-linear domain and the no hard budget domain.

Finally we introduce a *positive budget* condition. The domain includes only utility functions whose budgets are positive.

Definition 7. A domain $\mathcal{U}$ satisfies **positive budget** if for each $u_i \in \mathcal{U}$, $b_i > 0$.

Note that this conditions is about the restriction of a domain, while the first three conditions are about richness of a domain.

4.1 Strategy-proofness

In Definition 3, we specify no tie-breaking rule, that is, for each $u \in \mathcal{U}^n$ and each $i \in N$, when $\min\{v_i, b_i\} = \max\{\max_{j \neq i} \min\{v_j, b_j\}, r_i(u_{-i})\}$, it is possible that either $x_i(u) = 0$ or $x_i(u) = 1$. For some tie-breaking rule, a truncated Vickrey rule with endogenous reserve prices does not satisfy *strategy-proofness.*

To see this, let f be a truncated Vickrey rule with endogenous reserve prices r. Let $i \in N$, $u_{-i} \in \mathcal{U}^{n-1}$ and $r_i^*(u_{-i}) = \max\{\max_{j \neq i} \min\{v_j, b_j\}, r_i(u_{-i})\}$. Let $u_i, u_i' \in \mathcal{U}$ be such that $u_i(1, b_i) > 0$, $\min\{v_i, b_i\} = r_i^*(u_{-i})$, and $\min\{v_i', b_i'\} > r_i^*(u_{-i})$. By $u_i(1, b_i) > 0$, $v_i = \infty$. By $\min\{v_i', b_i'\} > r^*(u_{-i})$, $f_i(u_i', u_{-i}) = (1, r^*(u_{-i}))$. By $v_i = \infty$, $b_i = \min\{v_i, b_i\} = r_i^*(u_{-i})$, and so $f_i(u_i', u_{-i}) = (1, b_i)$. By $u_i(1, b_i) > 0$, $u_i(f_i(u_i', u_{-i})) > 0$. Thus, if $x_i(u) = 0$, then by $f_i(u) = (0, 0)$, $u_i(f_i(u_i', u_{-i})) > 0 = u_i(f_i(u))$. Hence, f violates *strategy-proofness.*

In the above case, if we set $x_i(u) = 1$, then we can avoid the strategical action of agent i to report u_i'. Thus, *strategy-proofness* requires that the agent with $v_i = \infty$ has a higher priority when ties exist. We formally define this tie-braking rule. We say that a truncated Vickrey rule f with endogenous reserve prices r has a **prioritized tie-breaking rule** if for each $u \in \mathcal{U}$ and each $i \in N$, if $\min\{v_i, b_i\} = \max\{\max_{j \neq i} \min\{v_j, b_j\}, r_i(u_{-i})\}$ and $v_i = \infty$, then $x_i(u) = 1$. The prioritized tie-breaking rule is shown to be a necessary and sufficient conditions to satisfy *strategy-proofness.*

Proposition 1. *Let $\mathcal{U} \subseteq \mathcal{U}^C$ satisfy unbounded truncated valuations. Let f on $\mathcal{U}^n$ be a truncated Vickrey rule with endogenous reserve prices r. Then, f satisfies strategy-proofness if and only if f has a prioritized tie-breaking rule.*

[3] To see this, let $a \in \mathbb{R}_+$. If $\mathcal{U}$ satisfies density, it is obviously rich. If $\mathcal{U}$ satisfies small compensation, let $u_i \in \mathcal{U}$ be such that $0 < c_i(a+1)-(a+1) < (a+2)-(a+1)$. Then, by $u_i(0, (a+1)-c_i(a+1)) = u_i(1, a+1)$ and $(a+1) - c_i(a+1) < 0$, $u_i(1, a+1) > 0$, and so $\min\{v_i, b_i\} \geq a+1 > a$. Thus, $\mathcal{U}$ is rich.

Our second result is a characterization of the rules satisfying *individual rationality*, *no subsidy for losers*, and *strategy-proofness*.[4]

Proposition 2. *Let $\mathcal{U} \subseteq \mathcal{U}^C$ satisfy unbounded truncated valuations. Then, a rule f on $\mathcal{U}^n$ satisfies individual rationality, no subsidy for losers, and strategy-proofness if and only for each $i \in N$, there is a price function $r_i : \mathcal{U}^{n-1} \to \mathbb{R} \cup \{\infty\}$ such that for each $u_{-i} \in \mathcal{U}^{n-1}$,*

(i-a) *for each $u_i \in \mathcal{U}$,*

$$x_i(u) = \begin{cases} 0 & \text{if } \min\{v_i, b_i\} < r_i(u_{-i}) \\ 1 & \text{if } \min\{v_i, b_i\} > r_i(u_{-i}) \end{cases},$$

(i-b) *for each $u_i \in \mathcal{U}$, if $\min\{v_i, b_i\} = r_i(u_{-i})$ and $v_i = \infty$, then $x_i(u) = 1$,*

(ii) *for each $u_i \in \mathcal{U}$,*

$$t_i(u) = \begin{cases} 0 & \text{if } x_i(u) = 0 \\ r_i(u_{-i}) & \text{if } x_i(u) = 1 \end{cases}.$$

4.2 Weak efficiency

First, we show that *efficiency* can be decomposed by three conditions.

Proposition 3. *Let $u \in \mathcal{U}^n$ and $z \in Z$. Then, z is efficient for u if and only if*

(i) *for each $i \in N$, $t_i \leq b_i$,*

(ii) *$\sum_{i \in N} x_i = 1$, and*

(iii) *there is no $z' \in Z$ such that $\sum_{i \in N} x'_i = \sum_{i \in N} x_i$ and it dominates z for u.*

For the conditions in Proposition 3, we call (i) **within budget**, (ii) **no wastage**, and (iii) **constrained efficiency**. We similarly define these conditions as those imposed on a rule.

- **Within budget**: For each $u \in \mathcal{U}^n$ and $i \in N$, $t_i(u) \leq b_i$.
- **No wastage**: For each $u \in \mathcal{U}^n$, $\sum_{i \in N} x_i(u) = 1$.
- **Constrained efficiency**: For each $u \in \mathcal{U}^n$, $f(u)$ is constrained efficient for u.

Corollary 1. *Let $\mathcal{U} \subseteq \mathcal{U}^C$. Then, a rule f on $\mathcal{U}^n$ satisfies efficiency if and only if it satisfies within budget, no wastage, and constrained efficiency.*

[4] Mukherjee (2014), Nisan (2007), and Sprumont (2013) show the similar results when the domain is quasi-linear and there is no budget constraint.

Similarly to *strategy-proofness*, for some tie-breaking rule, a truncated Vickrey rule with endogenous reserve prices may violate *constrained efficiency*.

To see this, let f be a truncated Vickrey rule with endogenous reserve prices r, and let $u \in \mathcal{U}^n$ and $i \in N$ be such that $v_i \neq \infty$ and $N^\infty(u) \neq \emptyset$. Let $r_i^*(u_{-i}) = \max\{\max_{j \neq i} \min\{v_j, b_j\}, r_i(u_{-i})\}$. Assume $x_i(u) = 1$. By $x_i(u) = 1$, i's truncated valuations is the highest, that is, $i \in N(u)$. Let $j \in N^\infty(u)$ be another agent whose truncated valuation is the highest. Then, $\min\{v_i, b_i\} = r_i^*(u_{-i}) \geq \min\{v_j, b_j\} = \min\{v_i, b_i\}$, and so $\min\{v_i, b_i\} = \min\{v_j, b_j\} = r_i^*(u_{-i})$. By $v_i \neq \infty$, $u_i(1, r_i^*(u_{-i})) = u_i(0, 0)$, and by $v_j = \infty$, $u_j(1, r_i^*(u_{-i})) > u_j(0, 0)$. By $f_i(u) = (1, r_i^*(u_{-i}))$ and $f_j(u) = (0, 0)$, reallocation between i and j dominates the original allocation $f(u)$. Thus, f violates *constrained efficiency*.

In the above illustration, if we exclude the agent i from the candidates who win the objects, then we can avoid the reallocation which undermines *constrained efficiency*. We formally define this tie-breaking rule. A truncated Vickrey rule with endogenous reserve prices r has an **exclusive tie-breaking rule** if for each $u \in \mathcal{U}^n$ and each $i \in N$, if $\min\{v_i, b_i\} = \max\{\max_{j \neq i} \min\{v_j, b_j\}, r_i(u_{-i})\}$, $v_i \neq \infty$, and $N^\infty(u) \neq \emptyset$, then $x_i(u) = 0$.

Proposition 4. *Let $\mathcal{U} \subseteq \mathcal{U}^C$. Let f on $\mathcal{U}^n$ be a truncated Vickrey rule with endogenous reserve prices r. Then, f satisfies constrained efficiency if and only if f has an exclusive tie-breaking rule.*

Next we characterize truncated Vickrey rule with endogenous reserve prices by *constrained efficiency*, *individual rationality*, *no subsidy for losers*, and *strategy-proofness*.

Theorem 1. *Let $\mathcal{U} \subseteq \mathcal{U}^C$ satisfy small compensation and positive budget. Then, a rule f on $\mathcal{U}^n$ satisfies constrained efficiency, individual rationality, no subsidy for losers, and strategy-proofness if and only if it is a truncated Vickrey rule with endogenous reserve prices and with an exclusive and prioritized tie-breaking rule.*

If the positive budget condition is violated, then there exists a rule that satisfies the four properties but not a truncated Vickrey rule.

Example 1. Let $\mathcal{U} \subseteq \mathcal{U}^n$ satisfy small compensation but violate positive budget. Let f on $\mathcal{U}^n$ be such that for each $i \in N$, there is $r_i : \mathcal{U}^{n-1} \to \mathbb{R} \cup \{\infty\}$ such that it satisfies the conditions in Proposition 2, and for each $u \in \mathcal{U}^n$, (i) if $\min\{v_i, b_i\} = r_i(u_{-i})$, $v_i \neq \infty$, and $N^\infty(u) \neq \emptyset$, then $x_i(u) = 0$ and (ii) if $\max_{j \neq i} \min\{v_j, b_j\} = 0$, then $r_i(u_{-i}) < \max_{j \neq i} \min\{v_j, b_j\} = 0$. Then, f satisfies *constrained efficiency*, *individual rationality*, *no subsidy for losers*, and *strategy-proofness*. However, by (ii), f is not a truncated Vickrey rule with endogenous reserve prices.

The final result shows that *no wastage* is incompatible with the four properties.

Proposition 5. *Let $\mathcal{U} \subseteq \mathcal{U}^C$ satisfy small compensation, strong density, and positive budget. Let f on $\mathcal{U}^n$ be a truncated Vickrey rule with endogenous reserve prices r and with an exclusive and prioritized tie-breaking rule. Then, f violates no wastage.*

Since *efficiency* implies both *no wastage* and *constrained efficiency*, it is incompatible with *individual rationality*, *no subsidy for losers*, and *strategy-proofness*.

Corollary 2. *Let $\mathcal{U} \subseteq \mathcal{U}^n$ satisfy small compensation, strong density, and positive budget. There is no rule on $\mathcal{U}^n$ that satisfies efficiency, individual rationality, no subsidy for losers, and strategy-proofness.*

4.3 Fairness

In general, a truncated Vickrey rule with endogenous reserve prices may not satisfy *weak envy-freeness for equals*. To see this, consider a simple case with $n = 2$. Let f be a truncated Vickrey rule with endogenous reserve prices r, and let $u \in \mathcal{U}^2$ be such that $u_1 = u_2$ and $u_1(1, b_1) = u_2(1, b_2) > 0$. By definition, $v_1 = v_2 = \infty$. Assume $x_1(u) = 0$ and $x_2(u) = 1$. Then, by definition $f_1(u) = (0, 0)$ and $f_2(u) = (1, \max\{\min\{v_1, b_1\}, r_2(u_1)\})$. By $v_1 = \infty$, $t_2(u) = \max\{b_1, r_2(u_1)\}$. If $b_1 \geq r_2(u_1)$, then by $u_1(1, b_1) > 0$ and $f_1(u) = (0, 0)$, $u_1(f_2(u)) = u_1(1, b_1) > 0 = u_1(f_1(u))$. Thus, f violates *weak envy-freeness for equals*.

This simple example gives us an intuitive reason why the truncated Vickrey rule does not work for *weak envy-freeness for equals*. In the above situation, since agent 2's reserve price is no more than his/her payment, that is, $r_2(u_1) \leq b_1 = t_2(u)$, the truncated Vickrey rule realizes so low payment for agent 2 that agent 1 envies agent 2. To handle this problem, it seems to be a good way to increase the agent 2's reserve price, that is, to set $r_2(u_1) > b_1$. Then, since agent 2's payment increases, agent 1 will no longer envy agent 2.

Formally, we define this condition. We say that a price function profile r is **non-negligible** if for each $i \in N$ and each $u_{-i} \in \mathcal{U}^{n-1}$, if there is $j \in N \setminus \{i\}$ such that $\min\{v_j, b_j\} = \max_{k \neq i} \min\{v_k, b_k\}$ and $v_j = \infty$, then $r_i(u_{-i}) > \max_{k \neq i} \min\{v_k, b_k\}$.

We show that the non-negligibility condition is a necessary and sufficient condition for truncated Vickrey rule with endogenous reserve prices and with a prioritized tie-breaking rule to satisfy *weak envy-freeness for equals*.

Theorem 2. *Let $\mathcal{U} \subseteq \mathcal{U}^C$ satisfy unbounded truncated valuations. Then, a rule f on $\mathcal{U}^n$ satisfies weak envy-freeness for equals, individual rationality, no subsidy for losers, and strategy-proofness if and only if it is a truncated Vickrey rule with non-negligible endogenous reserve prices r and with a prioritized tie-breaking rule.*

We show that the same characterization result holds even if we replace *weak envy-freeness for equals* with *equal treatment of equals* or *envy-freeness*.

Proposition 6. *Let $\mathcal{U} \subseteq \mathcal{U}^C$ satisfy unbounded truncated valuations. Let f on $\mathcal{U}^n$ be a truncated Vickrey rule with non-negligible endogenous reserve prices r. Then, f satisfies equal treatment of equals and envy-freeness.*

Since *equal treatment of equals* and *envy-freeness* imply *weak envy-freeness for equals*, by Theorem 2 and Proposition 6, we can get the following result.

Corollary 3. *Let $\mathcal{U} \subseteq \mathcal{U}^C$ satisfy unbounded truncated valuations. Then, the following three statements are equivalent.*

(i) *A rule f on $\mathcal{U}^n$ is a truncated Vickrey rule with non-negligible endogenous reserve prices r and with a prioritized tie-breaking rule.*

(ii) *A rule f on $\mathcal{U}^n$ satisfies equal treatment of equals, individual rationality, no subsidy for losers, and strategy-proofness.*

(iii) *A rule f on $\mathcal{U}^n$ satisfies envy-freeness, individual rationality, no subsidy for losers, and strategy-proofness.*

As for *anonymity in welfare*, we need an additional condition on reserve prices. A price function profile $r \in \mathcal{R}$ is **upper anonymous** if for each $u, u' \in \mathcal{U}^n$ and $i, j \in N$ such that $u_i = u'_j$, $u_j = u'_i$, and $u_{-\{i,j\}} = u'_{-\{i,j\}}$, when $r_i(u_{-i}) > \max_{k \neq i} \min\{v_k, b_k\}$, $r_i(u_{-i}) = r_j(u'_{-j})$.

Theorem 3. *Let $\mathcal{U} \subseteq \mathcal{U}^C$ satisfy density. Then, a rule f on $\mathcal{U}^n$ satisfies anonymity in welfare, individual rationality, no subsidy for losers, and strategy-proofness if and only if it is a truncated Vickrey rule with upper anonymous and non-negligible endogenous reserve prices r and with a prioritized tie-breaking rule.*

4.4 Independence

Given $u_i \in \mathcal{U}$ and δ, let v_i^δ be such that (i) if $u_i(1, b_i) \leq u_i(0, \delta)$, then $u_i(1, v_i^\delta) = u_i(0, \delta)$, and (ii) if $u_i(1, b_i) > u_i(0, \delta)$, then $v_i^\delta = \infty$. This is a valuation for u_i evaluated at $(0, \delta)$.

Definition 8. Given a price function profile r, a rule f on $\mathcal{R}^n$ is a **truncated Vickrey rule with endogenous reserve prices r and an entry fee δ** if for each $u \in \mathcal{U}^n$ and each $i \in N$,

$$x_i(u) = \begin{cases} 0 & \text{if } \min\{v_i^\delta, b_i\} < \max\{\max_{j \neq i} \min\{v_j^\delta, b_j\}, r_i(u_{-i})\} \\ 1 & \text{if } \min\{v_i^\delta, b_i\} > \max\{\max_{j \neq i} \min\{v_j^\delta, b_j\}, r_i(u_{-i})\} \end{cases},$$

and

$$t_i(u) = \begin{cases} \delta & \text{if } x_i(u) = 0 \\ \max\{\max_{j \neq i} \min\{v_j^\delta, b_j\}, r_i(u_{-i})\} & \text{if } x_i(u) = 1 \end{cases}.$$

We consider the independence of the properties. Let $\mathcal{U}$ satisfy small compensation and positive budget, and let f on $\mathcal{U}^n$ be a truncated Vickrey rule with endogenous reserve prices r and an entry fee δ.

- Dropping constrained efficiency: Assume f has a prioritized tie-breaking rule and $\delta = 0$, but the tie-breaking rule is not exclusive. Then, by Proposition 2 and Theorem 1, f satisfies *individual rationality*, *no subsidy for losers* and *strategy-proofness*, but not *constrained efficiency*.

- Dropping weak envy-freeness for equals: Assume f has a prioritized tie-breaking rule and $\delta = 0$, but r violates non-negligibility. Then, by Proposition 2 and Theorem 2, f satisfies *individual rationality*, *no subsidy for losers* and *strategy-proofness*, but not *weak envy-freeness for equals*.

The above rule f also violates other fairness conditions: *equal treatment of equals*, *envy freeness*, and *anonymity in welfare*.

- Dropping individual rationality: Assume f has an exclusive and prioritized tie-breaking rule, r satisfies non-negligibility, and $\delta > 0$. Then, f satisfies *constrained efficiency*, *weak envy-freeness for equals*, *no subsidy for losers* and *strategy-proofness*, but not *individual rationality*.

- Dropping no subsidy for losers: Assume f has an exclusive and prioritized tie-breaking rule, r satisfies non-negligibility, and $\delta < 0$. Then, f satisfies *constrained efficiency*, *weak envy-freeness for equals*, *individual rationality* and *strategy-proofness*, but not *no subsidy for losers*.

- Dropping strategy-proofness: Assume for some $u_i \in \mathcal{U}$, $v_i = \infty$.[5] Assume f has an exclusive tie-breaking rule, r satisfies non-negligibility and $\delta = 0$, but f has a non-prioritized tie-breaking rule. Then, by Propositions 1, 4, 6, f satisfies *constrained efficiency*, *weak envy-freeness for equals*, *individual rationality* and *no subsidy for losers*, but not *strategy-proofness*. When for each $u_i \in \mathcal{U}$, $v_i \neq \infty$, pay as bid rules satisfy all the properties except for *strategy-proofness*.

The above three results also holds for *equal treatment of equals* and *envy-freeness*, instead of *weak envy-freeness for equals*. Moreover, if r is upper anonymous, the same results holds for *anonymity in welfare*.

5 Conclusion

We consider the single-object allocation problem with hard budget constraints and income effects, and show that truncated Vickrey rules with endogenous reserve prices can be characterized by *constrained efficiency* or *weak envy-freeness for equals*, along with *individual rationality*, *no subsidy for losers*, and *strategy-proofness*. We leave the problem for the case of homogenous or heterogenous objects as an open question.

[5] If for each $u_i \in \mathcal{U}$, $v_i \neq \infty$, f always has a prioritized tie-breaking rule.

Appendix: proofs

Proposition 1. *Let $\mathcal{U} \subseteq \mathcal{U}^C$ satisfy unbounded truncated valuations. Let f on $\mathcal{U}^n$ be a truncated Vickrey rule with endogenous reserve prices r. Then, f satisfies strategy-proofness if and only if f has a prioritized tie-breaking rule.*

Proof. Only if: Assume f satisfies *strategy-proofness*. Let $u \in \mathcal{U}^n$ and $i \in N$ be such that $\min\{v_i, b_i\} = \max\{\max_{j \neq i} \min\{v_j, b_j\}, r_i(u_{-i})\}$ and $v_i = \infty$. Suppose $x_i(u) = 0$. By (V-ii), $f_i(u) = (0, 0)$. Let $r_i^*(u_{-i}) = \max\{\max_{j \neq i} \min\{v_j, b_j\}, r_i(u_{-i})\}$. By $\min\{v_i, b_i\} = r_i^*(u_{-i})$ and $v_i = \infty$, $u_i(1, r_i^*(u_{-i})) > u_i(0, 0)$. Let $u_i' \in \mathcal{U}$ be such that $\min\{v_i', b_i'\} > r_i^*(u_{-i})$. By (V-i) and (V-ii), $f_i(u_i', u_{-i}) = (1, r_i^*(u_{-i}))$. Thus, by $f_i(u) = (0, 0)$ and $u_i(1, r_i^*(u_{-i})) > u_i(0, 0)$, $u_i(f_i(u_i', u_{-i})) > u_i(f_i(u))$. However, this contradicts *strategy-proofness*.

If: Assume f has a prioritized tie-breaking rule. Suppose f violates *strategy-proofness*. Then, there are $u \in \mathcal{U}^n$, $i \in N$, and $u_i' \in \mathcal{U}$ such that $u_i(f_i(u_i', u_{-i})) > u_i(f_i(u))$. Let $r_i^*(u_{-i}) = \max\{\max_{j \neq i} \min\{v_j, b_j\}, r_i(u_{-i})\}$. By $u_i(f_i(u_i', u_{-i})) > u_i(f_i(u))$ and (V-ii), $x_i(u) \neq x_i(u_i', u_{-i})$. If $x_i(u_i', u_{-i}) = 0$, then $u_i(0, 0) > u_i(f_i(u))$, and so by $x_i(u) = 1$, $r_i^*(u_{-i}) = t_i(u) > \min\{v_i, b_i\}$. However, this contradicts (V-i). Hence, $x_i(u_i', u_{-i}) = 1$, and so $x_i(u) = 0$. By $u_i(f_i(u_i', u_{-i})) > u_i(f_i(u))$, $u_i(1, r_i^*(u_{-i})) > u_i(0, 0)$. By $x_i(u) = 0$ and (V-i), $\min\{v_i, b_i\} \leq r_i^*(u_{-i})$. By $u_i(1, r_i^*(u_{-i})) > u_i(0, 0)$, $r_i^*(u_{-i}) \leq \min\{v_i, b_i\}$, and so $\min\{v_i, b_i\} = r_i^*(u_{-i})$. By $u_i(1, r_i^*(u_{-i})) > u_i(0, 0)$ and $r_i^*(u_{-i}) = \min\{v_i, b_i\}$, $u_i(1, \min\{v_i, b_i\}) > u_i(0, 0)$, which implies $v_i = \infty$. However, by $x_i(u) = 0$ and $\min\{v_i, b_i\} = r_i^*(u_{-i})$, this contradicts that f has a prioritized tie-breaking rule. □

Lemma 1. *Let $\mathcal{U} \subseteq \mathcal{U}^C$ satisfy unbounded truncated valuations. A rule f on $\mathcal{U}^n$ satisfies individual rationality and strategy-proofness if and only if for each $i \in N$, there is a pair of price functions $\underline{r}_i : \mathcal{U}^{n-1} \to \mathbb{R} \cup \{\infty\}$ and $\overline{r}_i : \mathcal{U}^{n-1} \to \mathbb{R} \cup \{\infty\}$ such that for each $u_{-i} \in \mathcal{U}^{n-1}$,*

(i-L) $\underline{r}_i(u_{-i}) = \infty$ *if* $\mathcal{U}^L(u_{-i}) = \emptyset$, *and* $\underline{r}_i(u_{-i}) \leq 0$ *if* $\mathcal{U}^L(u_{-i}) \neq \emptyset$,

(i-W) $\overline{r}_i(u_{-i}) = \infty$ *if* $\mathcal{U}^W(u_{-i}) = \emptyset$, *and* $\overline{r}_i(u_{-i}) \leq \inf_{u_i \in \mathcal{U}^W(u_{-i})} \min\{v_i, b_i\}$ *if* $\mathcal{U}^W(u_{-i}) \neq \emptyset$,

(ii) *for each* $u_i \in \mathcal{U}$,

$$x_i(u) = \begin{cases} 0 & \text{if } u_i(0, \underline{r}_i(u_{-i})) > u_i(1, \overline{r}_i(u_{-i})) \\ 1 & \text{if } u_i(0, \underline{r}_i(u_{-i})) < u_i(1, \overline{r}_i(u_{-i})) \end{cases},$$

(iii) *for each* $u_i \in \mathcal{U}$,

$$t_i(u) = \begin{cases} \underline{r}_i(u_{-i}) & \text{if } x_i(u) = 0 \\ \overline{r}_i(u_{-i}) & \text{if } x_i(u) = 1 \end{cases}.$$

Proof. ONLY IF: Assume f satisfies *individual rationality* and *strategy-proofness*. Let $i \in N$ and $u_{-i} \in \mathcal{U}^{n-1}$.

First, we show that for each $u_i, u_i' \in \mathcal{U}$, if $x_i(u_i, u_{-i}) = x_i(u_i', u_{-i})$, then $t_i(u_i, u_{-i}) = t_i(u_i', u_{-i})$. Let $u_i, u_i' \in \mathcal{U}$ be such that $x_i(u_i, u_{-i}) = x_i(u_i', u_{-i})$. Suppose, without loss of generality, that $t_i(u_i, u_{-i}) < t_i(u_i', u_{-i})$. By *individual rationality*, $t_i(u_i', u_{-i}) \leq b_i'$. Hence, by $x_i(u_i, u_{-i}) = x_i(u_i', u_{-i})$ and $t_i(u_i, u_{-i}) < t_i(u_i', u_{-i}) \leq b_i'$, $u_i'(f_i(u_i, u_{-i})) > u_i'(f_i(u_i', u_{-i}))$. However, this contradicts *strategy-proofness*. Thus, $t_i(u_i, u_{-i}) = t_i(u_i', u_{-i})$.

Let $\underline{r}_i : \mathcal{U}^{n-1} \to \mathbb{R} \cup \{\infty\}$ and $\overline{r}_i : \mathcal{U}^{n-1} \to \mathbb{R} \cup \{\infty\}$ be such that for each $u_{-i} \in \mathcal{U}^{n-1}$,

$$\underline{r}_i(u_{-i}) = \begin{cases} t_i(u_i, u_{-i}) & \text{if there exists } u_i \in \mathcal{U}^L(u_{-i}) \\ \infty & \text{if } \mathcal{U}^L(u_{-i}) = \emptyset \end{cases},$$

$$\overline{r}_i(u_{-i}) = \begin{cases} t_i(u_i, u_{-i}) & \text{if there exists } u_i \in \mathcal{U}^W(u_{-i}) \\ \infty & \text{if } \mathcal{U}^W(u_{-i}) = \emptyset \end{cases}.$$

Note that, by the statement shown in the previous paragraph, these functions are uniquely defined and satisfy (iii).

Next, we show (i). If $\mathcal{U}^L(u_{-i}) = \emptyset$ and $\mathcal{U}^W(u_{-i}) = \emptyset$, then by definition, $\underline{r}_i(u_{-i}) = \overline{r}_i(u_{-i}) = \infty$. Assume $\mathcal{U}^L(u_{-i}) \neq \emptyset$ and $\mathcal{U}^W(u_{-i}) \neq \emptyset$. For each $u_i \in \mathcal{U}^L(u_{-i})$, by $f_i(u) = (0, \underline{r}_i(u_{-i}))$ and $u_i(f_i(u)) \geq 0$ (*individual rationality*), $\underline{r}_i(u_{-i}) \leq 0$. For each $u_i \in \mathcal{U}^W(u_{-i})$, by $f_i(u) = (1, \overline{r}_i(u_{-i}))$ and $u_i(f_i(u)) \geq 0$ (*individual rationality*), $\overline{r}_i(u_{-i}) \leq \min\{v_i, b_i\}$, and so $\overline{r}_i(u_{-i}) \leq \inf_{u_i \in \mathcal{U}^W(u_{-i})} \min\{v_i, b_i\}$.

Finally, we show (ii). Let $u_i \in \mathcal{U}$. First assume $u_i(0, \underline{r}_i(u_{-i})) > u_i(1, \overline{r}_i(u_{-i}))$. Suppose $x_i(u) = 1$. By (iii), $f_i(u) = (1, \overline{r}_i(u_{-i}))$. If $\mathcal{U}^L(u_{-i}) = \emptyset$, then by definition, $\underline{r}_i(u_{-i}) = \infty$, and so $u_i(0, \underline{r}_i(u_{-i})) = -\infty$. However, this contradicts $u_i(0, \underline{r}_i(u_{-i})) > u_i(1, \overline{r}_i(u_{-i}))$. Hence, $\mathcal{U}^L(u_{-i}) \neq \emptyset$. Let $u_i' \in \mathcal{U}^L(u_{-i})$. By (iii), $f_i(u_i', u_{-i}) = (0, \underline{r}_i(u_{-i}))$. By $u_i(0, \underline{r}_i(u_{-i})) > u_i(1, \overline{r}_i(u_{-i}))$, $f_i(u) = (1, \overline{r}_i(u_{-i}))$, and $f_i(u_i', u_{-i}) = (0, \underline{r}_i(u_{-i}))$, $u_i(f_i(u_i', u_{-i})) > u_i(f_i(u))$. However, this contradicts *strategy-proofness*. Hence, $x_i(u) = 0$.

Next assume $u_i(1, \overline{r}_i(u_{-i})) > u_i(0, \underline{r}_i(u_{-i}))$. Suppose $x_i(u) = 0$. By (iii), $f_i(u) = (0, \underline{r}_i(u_{-i}))$. If $\mathcal{U}^W(u_{-i}) = \emptyset$, then by definition, $\overline{r}_i(u_{-i}) = \infty$, and so $u_i(1, \overline{r}_i(u_{-i})) = -\infty$. However, this contradicts $u_i(1, \overline{r}_i(u_{-i})) > u_i(0, \underline{r}_i(u_{-i}))$. Hence, $\mathcal{U}^W(u_{-i}) \neq \emptyset$. Let $u_i' \in \mathcal{U}^W(u_{-i})$. By (iii), $f_i(u_i', u_{-i}) = (1, \overline{r}_i(u_{-i}))$. By $u_i(1, \overline{r}_i(u_{-i})) > u_i(0, \underline{r}_i(u_{-i}))$, $f_i(u) = (0, \underline{r}_i(u_{-i}))$, and $f_i(u_i', u_{-i}) = (1, \overline{r}_i(u_{-i}))$, $u_i(f_i(u_i', u_{-i})) > u_i(f_i(u))$. However, this contradicts *strategy-proofness*. Hence, $x_i(u) = 1$.

IF: Assume that for each $i \in N$, there exist $\underline{r}_i : \mathcal{U}^{n-1} \to \mathbb{R} \cup \{-\infty\}$ and $\overline{r}_i : \mathcal{U}^{n-1} \to \mathbb{R} \cup \{+\infty\}$ satisfying (i), (ii), and (iii).

Individual rationality: Let $u \in \mathcal{U}^n$ and $i \in N$. First assume $x_i(u) = 0$. By (iii), $f_i(u) = (0, \underline{r}_i(u_{-i}))$. By $\mathcal{U}^L(u_{-i}) \neq \emptyset$ and (i-L), $\overline{r}_i(u_{-i}) \leq 0$, and so $u_i(f_i(u)) \geq u_i(0, 0) = 0$. Next assume $x_i(u) = 1$. By (iii), $f_i(u) = (1, \overline{r}_i(u_{-i}))$. By $\mathcal{U}^W(u_{-i}) \neq \emptyset$ and (i-W), $\overline{r}_i(u_{-i}) \leq \min\{v_i, b_i\}$, and so $u_i(f_i(u)) \geq u_i(1, \min\{v_i, b_i\}) \geq 0$.

Strategy-proofness: Let $u \in \mathcal{U}^n$, $i \in N$ and $u_i' \in \mathcal{U}$. If $x_i(u) = x_i(u_i', u_{-i})$, then by definition, $f_i(u) = f_i(u_i', u_{-i})$, and so $u_i(f_i(u)) = u_i(f_i(u_i', u_{-i}))$. Thus, we assume $x_i(u) \neq x_i(u_i', u_{-i})$. If $x_i(u) = 0$ and $x_i(u_i', u_{-i}) = 1$, then by (ii), $u_i(0, \underline{r}_i(u_{-i})) \geq u_i(1, \overline{r}_i(u_{-i}))$, and by (iii), $f_i(u) = (0, \underline{r}_i(u_{-i}))$ and $f_i(u_i', u_{-i}) = (1, \overline{r}_i(u_{-i}))$. These expressions imply $u_i(f_i(u)) \geq u_i(f_i(u_i', u_{-i}))$. Similarly, if $x_i(u) = 1$ and $x_i(u_i', u_{-i}) = 0$, then $u_i(f_i(u)) = u_i(1, \overline{r}_i(u_{-i})) \geq u_i(0, \underline{r}_i(u_{-i})) = u_i(f_i(u_i', u_{-i}))$. □

Proposition 2. *Let $\mathcal{U} \subseteq \mathcal{U}^C$ satisfy unbounded truncated valuations. Then, a rule f on $\mathcal{U}^n$ satisfies individual rationality, no subsidy for losers, and strategy-proofness if and only for each $i \in N$, there is a price function $r_i : \mathcal{U}^{n-1} \to \mathbb{R} \cup \{\infty\}$ such that for each $u_{-i} \in \mathcal{U}^{n-1}$,*

(i-a) *for each $u_i \in \mathcal{U}$,*

$$x_i(u) = \begin{cases} 0 & \text{if } \min\{v_i, b_i\} < r_i(u_{-i}) \\ 1 & \text{if } \min\{v_i, b_i\} > r_i(u_{-i}) \end{cases},$$

(i-b) *for each $u_i \in \mathcal{U}$, if $\min\{v_i, b_i\} = r_i(u_{-i})$ and $v_i = \infty$, then $x_i(u) = 1$,*

(ii) *for each $u_i \in \mathcal{U}$,*

$$t_i(u) = \begin{cases} 0 & \text{if } x_i(u) = 0 \\ r_i(u_{-i}) & \text{if } x_i(u) = 1 \end{cases}.$$

Proof. ONLY IF: Assume f satisfies *individual rationality*, *no subsidy for losers*, and *strategy-proofness*. Let $i \in N$. By Lemma 1, there exist $\underline{r}_i : \mathcal{U}^{n-1} \to \mathbb{R} \cup \{\infty\}$ and $\overline{r}_i : \mathcal{U}^{n-1} \to \mathbb{R} \cup \{\infty\}$ such that they satisfy the conditions in the proposition. Let $u_{-i} \in \mathcal{U}^{n-1}$. If $\mathcal{U}^L(u_{-i}) \neq \emptyset$, by Lemma 1 (i-L), $\underline{r}_i(u_{-i}) \leq 0$. Thus, for each $u_i \in \mathcal{U}^L(u_{-i})$, by no-subsidy for losers, $\underline{r}_i(u_{-i}) = t_i(u) = 0$. Hence, (ii) holds.

Next we show (i-a) for $\overline{r}_i$. Let $u_i \in \mathcal{U}$. First assume $\min\{v_i, b_i\} < \overline{r}_i(u_{-i})$. Suppose $x_i(u) = 1$. Then, $f_i(u) = (1, \overline{r}_i(u_{-i}))$. By $\min\{v_i, b_i\} < \overline{r}_i(u_{-i}) = t_i(u)$, $u_i(0,0) > u_i(f_i(u))$. However, this contradicts *individual rationality*, and so $x_i(u) = 0$. Next assume $\min\{v_i, b_i\} > \overline{r}_i(u_{-i})$. Suppose $x_i(u) = 0$. By $\mathcal{U}^L(u_{-i}) \neq \emptyset$, $t_i(u) = \underline{r}_i(u_{-i}) = 0$. Thus, $f_i(u) = (0, 0)$. If $\mathcal{U}^W(u_{-i}) = \emptyset$, then by Lemma 1 (i-W), $\overline{r}_i(u_{-i}) = \infty$. However, this contradicts $\min\{v_i, b_i\} > \overline{r}_i(u_{-i})$, and so $\mathcal{U}^W(u_{-i}) \neq \emptyset$. Let $u_i' \in \mathcal{U}^W(u_{-i})$. Then, $f_i(u_i', u_{-i}) = (1, \overline{r}_i(u_{-i}))$. By $\min\{v_i, b_i\} > \overline{r}_i(u_{-i})$, $f_i(u_i', u_{-i}) = (1, \overline{r}_i(u_{-i}))$, and $f_i(u) = (0, 0)$, $u_i(f_i(u_i', u_{-i})) > u_i(f_i(u))$. However, this contradicts *strategy-proofness*, and so $x_i(u) = 1$.

Finally, we show (i-b) for $\overline{r}_i$. Let $u_i \in \mathcal{U}$ be such that $\min\{v_i, b_i\} = \overline{r}_i(u_{-i})$ and $v_i = \infty$. Suppose $x_i(u) = 0$. Then, $f_i(u) = (0, 0)$. By $v_i = \infty$, $\min\{v_i, b_i\} = b_i$ and $u_i(1, b_i) > 0$. By $b_i = \overline{r}_i(u_{-i})$ and $f_i(u) = (0, 0)$, $u_i(1, \overline{r}_i(u_{-i})) > u_i(f_i(u))$. Let $u_i' \in \mathcal{U}$ be such that $\min\{v_i', b_i'\} > \overline{r}_i(u_{-i})$. Then, by (i-a) and (ii), $f_i(u_i', u_{-i}) = (1, \overline{r}_i(u_{-i}))$. By $u_i(1, \overline{r}_i(u_{-i})) > u_i(f_i(u))$ and $f_i(u_i', u_{-i}) = (1, \overline{r}_i(u_{-i}))$, $u_i(f_i(u_i', u_{-i})) > u_i(f_i(u))$. However, this contradicts *strategy-proofness*. Hence, $x_i(u) = 1$.

IF: Assume that for each $i \in N$, f has a price function $r_i : \mathcal{U}^{n-1} \to \mathbb{R} \cup \{\infty\}$ satisfying (i) and (ii). By (ii), f satisfies *no subsidy for losers.* Thus, we show that f satisfies *individual rationality* and *strategy-proofness.* Let $\overline{r}_i : \mathcal{U}^{n-1} \to \mathbb{R} \cup \{\infty\}$ and $\overline{r}_i : \mathcal{U}^{n-1} \to \mathbb{R} \cup \{\infty\}$ be such that for each $u_{-i} \in \mathcal{U}^{n-1}$, $\overline{r}_i(u_{-i}) = r_i(u_{-i})$, and $\underline{r}_i(u_{-i}) = \infty$ if $\mathcal{U}^L(u_{-i}) = \emptyset$ and $\underline{r}_i(u_{-i}) = 0$ if $\mathcal{U}^L(u_{-i}) \neq \emptyset$. By Lemma 1, it suffices to show that $\overline{r}_i$ and $\underline{r}_i$ satisfy the conditions in the proposition. Let $i \in N$ and $u_{-i} \in \mathcal{U}^{n-1}$.

First we show Lemma 1 (i). By the definition of $\underline{r}_i$, (i-L) is obvious. We show (i-W). If $\mathcal{U}^W(u_{-i}) \neq \emptyset$, then for each $u_i \in \mathcal{U}^W(u_{-i})$, by $x_i(u) = 1$ and (i-a), $\overline{r}_i(u_{-i}) \leq \min\{v_i, b_i\}$, and so $\overline{r}_i(u_{-i}) \leq \inf_{u_i \in \mathcal{U}^W(u_{-i})} \min\{v_i, b_i\}$. Suppose $\mathcal{U}^W(u_{-i}) = \emptyset$ but $\overline{r}_i(u_{-i}) \neq \infty$. Let $u_i \in \mathcal{U}$ be such that $\min\{v_i, b_i\} > \overline{r}_i(u_{-i})$. By (i-a), $x_i(u) = 1$. However, this contradicts $\mathcal{U}^W(u_{-i}) = \emptyset$. Hence, if $\mathcal{U}^W(u_{-i}) = \emptyset$, then $\overline{r}_i(u_{-i}) = \infty$.

Next we show Lemma 1 (ii). Let $u_i \in \mathcal{U}$. First assume $u_i(0, \underline{r}_i(u_{-i})) > u_i(1, \overline{r}_i(u_{-i}))$. If $\mathcal{U}^L(u_{-i}) = \emptyset$, then $\underline{r}_i(u_{-i}) = \infty$, and so $u_i(0, \underline{r}_i(u_{-i})) = -\infty$. However, this contradicts $u_i(0, \underline{r}_i(u_{-i})) > u_i(1, \overline{r}_i(u_{-i}))$. Thus, $\mathcal{U}^L(u_{-i}) \neq \emptyset$, and so $\underline{r}_i(u_{-i}) = 0$. By $u_i(0, 0) > u_i(1, \overline{r}_i(u_{-i}))$, $\min\{v_i, b_i\} < \overline{r}_i(u_{-i})$. By (i-a), $x_i(u) = 0$. Next assume $u_i(0, \underline{r}_i(u_{-i})) < u_i(1, \overline{r}_i(u_{-i}))$. If $\mathcal{U}^L(u_{-i}) = \emptyset$, then by $\mathcal{U}^W(u_{-i}) = \mathcal{U}$, $x_i(u) = 1$. Hence, assume $\mathcal{U}^L(u_{-i}) \neq \emptyset$. By the definition of $\underline{r}_i$, $\underline{r}_i(u_{-i}) = 0$. By $u_i(0, 0) < u_i(1, \overline{r}_i(u_{-i}))$, $\min\{v_i, b_i\} > \overline{r}_i(u_{-i})$, or $\min\{v_i, b_i\} = \overline{r}_i(u_{-i})$ and $v_i = \infty$. Hence, by (i-a) and (i-b), $x_i(u) = 1$.

Finally we show Lemma 1 (iii). Since $\underline{r}_i(u_{-i}) = 0$ if $\mathcal{U}^L(u_{-i}) \neq \emptyset$, the result follows from (ii). □

Proposition 3. *Let $u \in \mathcal{U}^n$ and $z \in Z$. Then, z is efficient for u if and only if*

(i) *for each $i \in N$, $t_i \leq b_i$,*

(ii) $\sum_{i \in N} x_i = 1$, *and*

(iii) *there is no $z' \in Z$ such that $\sum_{i \in N} x'_i = \sum_{i \in N} x_i$ and it dominates z for u.*

Proof. ONLY IF: Assume z is efficient for u. Since (iii) is obvious, we show (i) and (ii).

First we show (i). Suppose there is $i \in N$ such that $t_i > b_i$. Then, for $\varepsilon > 0$, $u_i(x_i, t_i + \varepsilon) = u_i(x_i, t_i)$. Thus, $((x_i, t_i + \varepsilon), z_{-i})$ dominates z for u, a contradiction.

Next we show (ii). Suppose $\sum_{i \in N} x_i = 0$. Let $j \in N$. By $\sum_{i \in N} x_i = 0$, $x_j = 0$. By (i), $t_j \leq b_j$, and so $u_j(1, t_j) > u_j(z_j)$. Hence, $((1, t_j), z_{-j})$ dominates z for u, a contradiction.

IF: Assume (i), (ii), and (iii) hold. Suppose z is not *efficient* for u. Then, there is $z' \in Z$ such that (a) for each $i \in N$, $u_i(z'_i) \geq u_i(z_i)$, (b) $\sum_{i \in N} t'_i \geq \sum_{i \in N} t_i$, and (c) at least one inequality holds strictly.

By (iii), $\sum_{i \in N} x'_i \neq \sum_{i \in N} x_i$. By (ii), $\sum_{i \in N} x_i = 1$, and so $\sum_{i \in N} x'_i = 0$. By $\sum_{i \in N} x_i = 1$, there is $i \in N$ such that $x_i = 1$. By $\sum_{j \in N} x'_j = 0$, $x'_i = 0$. By $t_i \leq b_i$, $u_i(z_i) = u_i(0, t_i - c_i(t_i))$. By $x'_i = 0$ and $u_i(z'_i) \geq u_i(z_i) = u_i(0, t_i - c_i(t_i))$, $t'_i \leq t_i - c_i(t_i)$. For each $j \in N \setminus \{i\}$, by $x'_j = x_j = 0$ and $u_j(z'_j) \geq u_j(z_j)$, $t'_j \leq t_j$. Thus, by $c_i(t_i) > 0$, $\sum_{j \in N} t'_j \leq (t_i - c_i(t_i)) + \sum_{j \neq i} t_j < \sum_{j \in N} t_j$. However, this contradicts (b). □

Lemma 2. *Let $u \in \mathcal{U}^n$ and $z \in Z$ be such that z satisfies within budget for u. Then, z is constrained efficient for u if and only if when there is $i \in N$ such that $x_i = 1$, for each $j \in N \setminus \{i\}$, $u_j(z_j) \geq u_j(1, t_j + c_i(t_i))$.*

Proof. ONLY IF: Assume there is $i \in N$ such that $x_i = 1$. Let $j \in N \setminus \{i\}$. By $x_i = 1$, $x_j = 0$. Suppose $u_j(1, t_j + c_i(t_i)) > u_j(z_j)$. By the definition of c_i, $u_i(0, t_i - c_i(t_i)) \geq u_i(z_i)$. Thus, $((0, t_i - c_i(t_i)), (1, t_j + c_i(t_i)), z_{-\{i,j\}})$ dominates z for u, a contradiction.

IF: Assume that if there is $i \in N$ such that $x_i = 1$, then for each $j \in N \setminus \{i\}$, $u_j(z_j) \geq u_j(1, t_j + c_i(t_i))$. Suppose z is not *constrained efficiency* for u. Then, there is $z' \in Z$ such that (i) $\sum_{i \in N} x'_i = \sum_{i \in N} x_i$, (ii) for each $i \in N$, $u_i(z'_i) \geq u_i(z_i)$, (iii) $\sum_{i \in N} t'_i \geq \sum_{i \in N} t_i$, and (iv) at least one inequality holds strictly.

First we show for some $i \in N$, $x'_i \neq x_i$. Suppose for each $i \in N$, $x'_i = x_i$. For each $i \in N$, by $t_i \leq b_i$, $x'_i = x_i$, and $u_i(z'_i) \geq u_i(z_i)$, $t'_i \leq t_i$. Thus, by $\sum_{i \in N} t'_i \geq \sum_{i \in N} t_i$, for each $i \in N$, $t'_i = t_i$. Hence, for each $i \in N$, $u_i(z'_i) = u_i(z_i)$ and $\sum_{i \in N} t'_i = \sum_{i \in N} t_i$. However, these equalities contradict (iv).

Since for some $i \in N$, $x'_i \neq x_i$, $\sum_{i \in N} x'_i = 1$ or $\sum_{i \in N} x_i = 1$. Hence, by (i), $\sum_{i \in N} x'_i = \sum_{i \in N} x_i = 1$.

Finally, we derive a contradiction. Since for some $i \in N$, $x'_i \neq x_i$ and $\sum_{i \in N} x_i = \sum_{i \in N} x'_i = 1$, there are $i, j \in N$ such that $(x_i, x_j) = (1, 0)$ and $(x'_i, x'_j) = (0, 1)$. By $u_i(z'_i) \geq u_i(z_i) = u_i(0, t_i - c_i(t_i))$ and $x'_i = 0$, $t'_i \leq t_i - c_i(t_i)$. By $t_j \leq b_j$ and $u_j(z'_j) \geq u_j(z_j)$, $t'_j \leq b_j$. By (ii) and the assumption, $u_j(z'_j) \geq u_j(z_j) \geq u_j(1, t_j + c_i(t_i))$, and so by $t_j \leq b_j$, $t'_j \leq b_j$, and $x'_j = 1$, $t'_j \leq t_j + c_i(t_i)$. For each $k \in N \setminus \{i, j\}$, by $x'_k = x_k = 0$ and $u_k(z'_k) \geq u_k(z_k)$, $t'_k \leq t_k$. Hence, by $\sum_{k \in N} t'_k \geq \sum_{k \in N} t_k$, $t'_i = t_i - c_i(t_i)$, $t'_j = t_j + c_i(t_i)$, and for each $k \in N \setminus \{i, j\}$, $t'_k = t_k$. Thus, for each $k \in N$, $u_k(z'_k) = u_k(z_k)$ and $\sum_{k \in N} t'_k = \sum_{k \in N} t_k$. However, these equalities contradict (iv). □

Proposition 4. *Let $\mathcal{U} \subseteq \mathcal{U}^C$. Let f on $\mathcal{U}^n$ be a truncated Vickrey rule with endogenous reserve prices r. Then, f satisfies constrained efficiency if and only if f has an exclusive tie-breaking rule.*

Proof. ONLY IF: Assume f satisfies *constrained efficiency.* Let $u \in \mathcal{U}^n$ and $i \in N$ be such that $\min\{v_i, b_i\} = \max\{\max_{j \neq i} \min\{v_j, b_j\}, r_i(u_{-i})\}$, $v_i \neq \infty$, and $N^\infty(u) \neq \emptyset$. Suppose $x_i(u) = 1$. Let $r^*_i(u_{-i}) = \max\{\max_{j \neq i} \min\{v_j, b_j\}, r_i(u_{-i})\}$. By $\min\{v_i, b_i\} = r^*_i(u_{-i})$ and $v_i \neq \infty$, $i \in N(u) \setminus N^\infty(u)$. Let $j \in N^\infty(u)$. By $i \in N(u) \setminus N^\infty(u)$, $i \neq j$. By $i, j \in N(u)$, $\min\{v_i, b_i\} = \min\{v_j, b_j\}$. By $\min\{v_i, b_i\} = \min\{v_j, b_j\}$ and $i \neq j$, $\min\{v_i, b_i\} \geq r^*_i(u_{-i}) \geq \min\{v_j, b_j\} = \min\{v_i, b_i\}$, and so, $\min\{v_i, b_i\} = \min\{v_j, b_j\} = r^*_i(u_{-i})$. By $r^*_i(u_{-i}) = \min\{v_i, b_i\}$ and (V-ii), $t_i(u) = \min\{v_i, b_i\}$. By $t_i(u) = \min\{v_i, b_i\}$ and $v_i \neq \infty$, $u_i(f_i(u)) = u_i(0, 0)$. By $t_i(u) \leq b_i$, $u_i(0, t_i(u) - c_i(t_i(u))) = u_i(f_i(u))$, and so by $u_i(f_i(u)) = u_i(0, 0)$, $c_i(t_i(u)) = t_i(u)$. By $t_i(u) = \min\{v_j, b_j\}$, $c_i(t_i(u)) = \min\{v_j, b_j\}$. By $j \in N^\infty(u)$, $v_j = \infty$, and so $u_j(1, \min\{v_j, b_j\}) > 0$. Thus, by $c_i(t_i(u)) = \min\{v_j, b_j\}$ and $f_j(u) = (0, 0)$, $u_j(1, t_j(u) + c_i(t_i(u))) > u_j(f_j(u))$. However, by Lemma 2, this contradicts *constrained efficiency.*

IF: Assume f has an exclusive tie-breaking rule. Suppose f violates *constrained efficiency*. Then, by Lemma 2, there are $u \in \mathcal{U}^n$ and $i, j \in N$ with $i \neq j$ such that $x_i(u) = 1$ and $u_j(1, t_j(u) + c_i(t_i(u))) > u_j(f_j(u))$. Let $r_i^*(u_{-i}) = \max\{\max_{j \neq i} \min\{v_j, b_j\}, r_i(u_{-i})\}$. By (V-i) and $i \neq j$, $\min\{v_i, b_i\} \geq r_i^*(u_{-i}) \geq \min\{v_j, b_j\}$. By $\min\{v_i, b_i\} \geq r_i^*(u_{-i}) = t_i(u)$, $u_i(f_i(u)) \geq u_i(0, 0)$. By $t_i(u) \leq b_i$, $u_i(0, t_i(u) - c_i(t_i(u))) = u_i(f_i(u))$. By $u_i(0, t_i(u) - c_i(t_i(u))) = u_i(f_i(u)) \geq u_i(0, 0)$, $c_i(t_i(u)) \geq t_i(u)$. By $u_j(1, t_j(u) + c_i(t_i(u))) > u_j(f_j(u))$ and $f_j(u) = (0, 0)$, $c_i(t_i(u)) \leq \min\{v_j, b_j\}$. Hence, $\min\{v_j, b_j\} \geq c_i(t_i(u)) \geq t_i(u) \geq \min\{v_j, b_j\}$, and so $c_i(t_i(u)) = t_i(u) = \min\{v_j, b_j\}$. By $c_i(t_i(u)) = t_i(u)$, $u_i(0, 0) = u_i(f_i(u))$, and so $\min\{v_i, b_i\} = t_i(u)$ and $v_i \neq \infty$. By $c_i(t_i(u)) = \min\{v_j, b_j\}$, $u_j(1, t_j(u) + c_i(t_i(u))) > u_j(f_j(u))$, and $f_j(u) = (0, 0)$, $v_j = \infty$. By $\min\{v_j, b_j\} = t_i(u) = \min\{v_i, b_i\}$ and $i \in N(u)$, $j \in N(u)$. By $v_j = \infty$, $j \in N^\infty(u)$. However, by $x_i(u) = 1$, $\min\{v_i, b_i\} = r_i^*(u_{-i})$, and $v_i \neq \infty$, this contradicts that f has an exclusive tie-breaking rule. □

Theorem 1. *Let $\mathcal{U} \subseteq \mathcal{U}^C$ satisfy small compensation and positive budget. Then, a rule f on $\mathcal{U}^n$ satisfies constrained efficiency, individual rationality, no subsidy for losers, and strategy-proofness if and only if it is a truncated Vickrey rule with endogenous reserve prices and with an exclusive and prioritized tie-breaking rule.*

Proof. Since "if" part follows from Propositions 2 and 4, we only show "only if" part. Assume f satisfies *constrained efficiency*, *individual rationality*, *no subsidy for losers*, and *strategy-proofness*. By Proposition 2, for each $i \in N$, there is a price function $r_i : \mathcal{U}^{n-1} \to \mathbb{R} \cup \{\infty\}$ satisfying the conditions in the proposition.

By Propositions 1 and 4, to prove the result, it suffices to show that f is a truncated Vickrey rule with endogenous reserve prices r, that is, for each $i \in N$ and $u_{-i} \in \mathcal{U}^{n-1}$, $r_i(u_{-i}) \geq \max_{j \neq i} \min\{v_j, b_j\}$.

Suppose there are $i \in N$ and $u_{-i} \in \mathcal{U}^{n-1}$ such that $r_i(u_{-i}) < \max_{j \neq i} \min\{v_j, b_j\}$. Let $j \in N \setminus \{i\}$ be such that $\min\{v_j, b_j\} = \max_{k \neq i} \min\{v_k, b_k\}$. Let $u_i \in \mathcal{U}$ be such that $r_i(u_{-i}) < c_i(r_i(u_{-i})) < \min\{v_j, b_j\}$.[6]

First we show $x_i(u) = 1$. If $r_i(u_{-i}) > b_i$, then by $b_i \geq 0$, $c_i(r_i(u_{-i})) = r_i(u_{-i}) - b_i \leq r_i(u_{-i})$. However, this contradicts $r_i(u_{-i}) < c_i(r_i(u_{-i}))$, and so $r_i(u_{-i}) \leq b_i$. By $r_i(u_{-i}) \leq b_i$, $u_i(0, r_i(u_{-i}) - c_i(r_i(u_{-i}))) = u_i(1, r_i(u_{-i}))$. By $r_i(u_{-i}) < c_i(r_i(u_{-i}))$, $u_i(1, r_i(u_{-i})) > u_i(0, 0)$. By $u_i(1, r_i(u_{-i})) > u_i(0, 0)$, $r_i(u_{-i}) < \min\{v_i, b_i\}$, or $r_i(u_{-i}) = \min\{v_i, b_i\}$ and $v_i = \infty$. Thus, by Proposition 2 (i-a) and (i-b), $x_i(u) = 1$.

Finally we derive a contradiction. By $x_i(u) = 1$ and Proposition 2 (ii), $f_i(u) = (1, r_i(u_{-i}))$. By $t_i(u) = r_i(u_{-i})$ and $c_i(r_i(u_{-i})) < \min\{v_j, b_j\}$, $u_j(1, c_i(t_i(u))) > 0$. By $x_i(u) = 1$, $x_j(u) = 0$, and so $f_j(u) = (0, 0)$. Thus, by $u_j(1, c_i(t_i(u))) > 0$ and $f_j(u) = (0, 0)$, $u_j(1, t_j(u) + c_i(t_i(u))) > u_j(f_j(u))$. However, by Lemma 2, this contradicts *constrained efficiency*. □

Proposition 5. *Let $\mathcal{U} \subseteq \mathcal{U}^C$ satisfy small compensation, strong density, and positive budget. Let f on $\mathcal{U}^n$ be a truncated Vickrey rule with endogenous reserve prices r and with an exclusive and prioritized tie-breaking rule. Then, f violates no wastage.*

[6] By small compensation, u_i exists.

Proof. Let $u \in \mathcal{U}^n$ be such that for each $i, j \in N$, $u_i = u_j$ and $v_i = \infty$. Given $i \in N$ and $\tilde{u}_{-i} \in \mathcal{U}^{n-1}$, let $r_i^*(\tilde{u}_{-i}) = \max\{\max_{j \neq i} \min\{\tilde{v}_j, \tilde{b}_j\}, r_i(\tilde{u}_{-i})\}$. For each $i \in N$, by $v_i = \infty$, $\min\{v_i, b_i\} \geq r_i^*(u_{-i})$ if and only if $x_i(u) = 1$. If for each $i \in N$, $\min\{v_i, b_i\} < r_i^*(u_{-i})$, then $\sum_{i \in N} x_i(u) = 0$, and so f violates *no wastage*. Assume there is $i \in N$ such that $\min\{v_i, b_i\} \geq r_i^*(u_{-i})$. Then, $x_i(u) = 1$. Let $j \in N \setminus \{i\}$. By $x_i(u) = 1$, $x_j(u) = 0$, and so $\min\{v_j, b_j\} < r_j^*(u_{-j})$. Let $u'_j \in \mathcal{U}$ be such that $\min\{v_j, b_j\} < \min\{v'_j, b'_j\} < r_j^*(u_{-j})$ and $v'_j = \infty$.[7] By $\min\{v'_j, b'_j\} < r_j^*(u_{-j})$, $x_j(u'_j, u_{-j}) = 0$. For each $k \in N \setminus \{j\}$, by $\min\{v_k, b_k\} = \min\{v_j, b_j\} < \min\{v'_j, b'_j\} \leq r_k^*(u'_j, u_{-\{k,j\}})$, $x_k(u'_j, u_{-j}) = 0$. Hence, f violates *no wastage*. □

Theorem 2. *Let $\mathcal{U} \subseteq \mathcal{U}^C$ satisfy unbounded truncated valuations. Then, a rule f on $\mathcal{U}^n$ satisfies weak envy-freeness for equals, individual rationality, no subsidy for losers, and strategy-proofness if and only if it is a truncated Vickrey rule with some non-negligible endogenous reserve prices r and with a prioritized tie-breaking rule.*

Proof. Only if: Let f satisfy *weak envy-freeness for equals*, *individual rationality*, *no subsidy for losers*, and *strategy-proofness*. By Proposition 2, for each $i \in N$, there is some price function $r_i : \mathcal{U}^{n-1} \to \mathbb{R} \cup \{\infty\}$ satisfying the conditions in the proposition.

First, we show that for each $i \in N$ and each $u_{-i} \in \mathcal{U}^{n-1}$, $r_i(u_{-i}) \geq \max_{j \neq i} \min\{v_j, b_j\}$. Let $i \in N$ and $u_{-i} \in \mathcal{U}^{n-1}$. Suppose that for some $j \in N \setminus \{i\}$, $\min\{v_j, b_j\} > r_i(u_{-i})$. By $r_i(u_{-i}) < \min\{v_j, b_j\}$, $u_j(1, r_i(u_{-i})) > 0$. Let $u_i \in \mathcal{U}$ be such that $u_i = u_j$. Then, by $\min\{v_i, b_i\} = \min\{v_j, b_j\} > r_i(u_{-i})$, Proposition 2 (i-a) and (ii), $f_i(u) = (1, r_i(u_{-i}))$. By $x_i(u) = 1$, $x_j(u) = 0$, and so by Proposition 2 (ii), $f_j(u) = (0, 0)$. Thus, by $u_j(1, r_i(u_{-i})) > 0$ and $f_i(u) = (1, r_i(u_{-i}))$, $u_j(f_i(u)) > u_j(f_j(u))$. However, by $u_j = u_i$, this contradicts *weak envy-freeness for equals*. Hence, $r_i(u_{-i}) \geq \max_{j \neq i} \min\{v_j, b_j\}$.

For each $i \in N$ and each $u_{-i} \in \mathcal{U}^{n-1}$, by $r_i(u_{-i}) \geq \max_{j \neq i} \min\{v_j, b_j\}$, $r_i(u_{-i}) = \max\{\max_{j \neq i} \min\{v_j, b_j\}, r_i(u_{-i})\}$. Hence, by Proposition 2, f is a truncated Vickrey rule with endogenous reserve prices r and with a prioritized tie-breaking rule.

Finally, we show that r is non-negligible. Let $i \in N$ and $u_{-i} \in \mathcal{U}^{n-1}$, and let $j \in N$ be such that $\min\{v_j, b_j\} = \max_{k \neq i} \min\{v_k, b_k\}$ and $v_j = \infty$. Suppose $r_i(u_{-i}) \leq \max_{k \neq i} \min\{v_k, b_k\}$. Let $r_i^*(u_{-i}) = \max\{\max_{k \neq i} \min\{v_k, b_k\}, r_i(u_{-i})\}$. Then, by $\min\{v_j, b_j\} = \max_{k \neq i} \min\{v_k, b_k\} \geq r_i(u_{-i})$, $r_i^*(u_{-i}) = \min\{v_j, b_j\}$. Let $u_i \in \mathcal{U}$ be such that $u_i = u_j$. By $\min\{v_j, b_j\} = r_i^*(u_{-i})$, $v_j = \infty$, and $u_i = u_j$, $\min\{v_i, b_i\} = r_i^*(u_{-i})$ and $v_i = \infty$. Thus, a prioritized tie-breaking rule requires $x_i(u) = 1$. By Proposition 2 (ii), $t_i(u) = r_i^*(u_{-i}) = \min\{v_j, b_j\}$. By $v_j = \infty$, $f_i(u) = (1, b_j)$. By $x_i(u) = 1$, $x_j(u) = 0$. By $x_j(u) = 0$ and Proposition 2 (ii), $f_j(u) = (0, 0)$. By $v_j = \infty$, $u_j(1, b_j) > 0$. Thus, by $f_i(u) = (1, b_j)$ and $f_j(u) = (0, 0)$, $u_j(f_i(u)) > u_j(f_j(u))$. However, this contradicts *weak envy-freeness for equals*.

If: Let f be a truncated Vickrey rule with non-negligible endogenous reserve prices r and with a prioritized tie-breaking rule. Since f satisfies all the conditions in Proposition

[7]By strong density, u'_j exists.

2, it satisfies *individual rationality*, *no subsidy for losers*, and *strategy-proofness*. Hence, we show that f satisfies *weak envy-freeness for equals*.

Let $u \in \mathcal{U}$ and $i, j \in N$ be such that $u_i = u_j$, $x_i(u) = 0$ and $x_j(u) = 1$. By $x_j(u) = 1$, $i \neq j$ and (V-i), $\min\{v_j, b_j\} \geq \max\{\max_{k \neq j} \min\{v_k, b_k\}, r_j(u_{-j})\} \geq \min\{v_i, b_i\}$. By $u_i = u_j$, the above inequalities hold with equality, and so $\min\{v_i, b_i\} = \max_{k \neq j} \min\{v_k, b_k\} \geq r_j(u_{-j})$. Thus, by non-negligibility, $v_i \neq \infty$, and so by definition, $v_i \leq b_i$. Hence, $\max\{\max_{k \neq j} \min\{v_k, b_k\}, r_j(u_{-j})\} = v_i$. By $x_i(u) = 0$, $x_j(u) = 1$, and (V-ii), $f_i(u) = (0, 0)$ and $f_j(u) = (1, v_i)$. Thus, by the definition of v_i, $u_i(f_i(u)) = u_i(f_j(u))$. □

Proposition 6. *Let $\mathcal{U} \subseteq \mathcal{U}^C$ satisfy unbounded truncated valuations. Let f on $\mathcal{U}^n$ be a truncated Vickrey rule with non-negligible endogenous reserve prices r. Then, f satisfies equal treatment of equals and envy-freeness.*

Proof. Since *envy-freeness* implies *equal treatment of equals*, we only show *envy-freeness*. Let $u \in \mathcal{U}^n$ and $i, j \in N$. If $x_i(u) = x_j(u) = 0$, by $f_i(u) = f_j(u) = (0, 0)$, $u_i(f_i(u)) = u_i(f_j(u))$. If $x_i(u) = 1$ and $x_i(u) = 0$, then by $f_j(u) = (0, 0)$ and *individual rationality*, $u_i(f_i(u)) \geq u_i(f_j(u))$. Assume $x_i(u) = 0$ and $x_j(u) = 1$. Given $k \in N$, let $r_k^*(u_{-k}) = \max\{\max_{h \neq k} \min\{v_h, b_h\}, r_k(u_{-k})\}$. By $x_j(u) = 1$, $f_j(u) = (1, r_j^*(u_{-j}))$. By $i \neq j$, $\min\{v_i, b_i\} \leq r_j^*(u_{-j})$. If $\min\{v_i, b_i\} < r_j^*(u_{-j})$, by $f_i(u) = (0, 0)$ and $f_j(u) = (1, r_j^*(u_{-j}))$, $u_i(f_i(u)) > u_i(f_j(u))$. Assume $\min\{v_i, b_i\} = r_j^*(u_{-j})$. By $\min\{v_i, b_i\} = r_j^*(u_{-j})$, $\min\{v_i, b_i\} = \max_{k \neq j} \min\{v_k, b_k\}$. By $\min\{v_i, b_i\} = r_j^*(u_{-j}) \geq r_j(u_{-j})$ and non-negligibility, $v_i \neq \infty$, and so $u_i(1, \min\{v_i, b_i\}) = u_i(0, 0)$. Thus, by $f_i(u) = (0, 0)$, $f_j(u) = (1, r_j^*(u_j))$, and $r_j^*(u_{-j}) = \min\{v_i, b_i\}$, $u_i(f_i(u)) = u_i(f_j(u))$. □

Theorem 3. *Let $\mathcal{U} \subseteq \mathcal{U}^C$ satisfy density. Then, a rule f on $\mathcal{U}^n$ satisfies anonymity in welfare, individual rationality, no subsidy for losers, and strategy-proofness if and only if it is a truncated Vickrey rule with upper anonymous and non-negligible endogenous reserve prices r and with a prioritized tie-breaking rule.*

Proof. ONLY IF: Let f satisfy *anonymity in welfare*, *individual rationality*, *no subsidy for losers*, and *strategy-proofness*. By Theorem 2, f is a truncated Vickrey rule with non-negligible reserve prices r and with a prioritized tie-breaking rule. Thus, we show r is upper anonymous. Let $u, u' \in \mathcal{U}^n$ and $i, j \in N$ be such that $u_i = u'_j$, $u_j = u'_i$, $u_{-\{i,j\}} = u'_{-\{i,j\}}$, and $r_i(u_{-i}) > \max_{k \neq i} \min\{v_k, b_k\}$. Suppose $r_i(u_{-i}) \neq r_j(u'_{-j})$. Given $k \in N$ and $\tilde{u}_{-k} \in \mathcal{U}^{n-1}$, let $r_k^*(\tilde{u}_{-k}) = \max\{\max_{h \neq k} \min\{\tilde{v}_h, \tilde{b}_h\}, r_k(\tilde{u}_{-k})\}$.

First, we show $r_i^*(u_{-i}) \neq r_j^*(u'_{-j})$. By $r_i(u_{-i}) > \max_{k \neq i} \min\{v_k, b_k\}$, $r_i^*(u_{-i}) = r_i(u_{-i})$. If $r_j(u'_{-j}) \leq \max_{k \neq j} \min\{v'_k, b'_k\}$, then by $r_i^*(u_{-i}) = r_i(u_{-i}) > \max_{k \neq i} \min\{v_k, b_k\}$ and $r_j^*(u'_{-j}) = \max_{k \neq j} \min\{v'_k, b'_k\} = \max_{k \neq i} \min\{v_k, b_k\}$, $r_i^*(u_{-i}) > r_j^*(u'_{-j})$. If $r_j(u'_{-j}) > \max_{k \neq j} \min\{v'_k, b'_k\}$, then by $r_i(u_{-i}) \neq r_j(u'_{-j})$, $r_i^*(u_{-i}) = r_i(u_{-i})$, and $r_j^*(u'_{-j}) = r_j(u'_{-j})$, $r_i^*(u_{-i}) \neq r_j^*(u'_{-j})$.

Next, we derive a contradiction. Without loss of generality, assume $r_i^*(u_{-i}) > r_j^*(u'_{-j})$. Let $\tilde{u}_i = \tilde{u}'_j \in \mathcal{U}$ be such that $r_i^*(u_{-i}) > \min\{\tilde{v}_i, \tilde{b}_i\} = \min\{\tilde{v}'_j, \tilde{b}'_j\} > r_j^*(u'_{-j})$.[8] Then, by

[8] By density, $\tilde{u}_i$ exists.

definition, $f_i(\tilde{u}_i, u_{-i}) = (0,0)$ and $f_j(\tilde{u}'_j, u'_{-j}) = (1, r^*_j(u'_{-j}))$. By $\min\{\tilde{v}_i, \tilde{b}_i\} > r^*_j(u'_{-j})$, $\tilde{u}_i(1, r^*_j(u'_{-j})) > \tilde{u}_i(0,0)$, and so $\tilde{u}_i(f_j(\tilde{u}'_j, u_{-j})) > \tilde{u}_i(f_i(\tilde{u}_i, u_{-i}))$. However, this contradicts *anonymity in welfare.*

IF: Let f be a truncated Vickrey rule with upper anonymous and non-negligible endogenous reserve prices r and with a prioritized tie-breaking rule. By Theorem 2, f satisfies *individual rationality*, *no subsidy for losers*, and *strategy-proofness*. Thus, we show f satisfies *anonymity in welfare.*

Let $u, u' \in \mathcal{U}^n$ and $i, j \in N$ be such that $u_i = u'_j$, $u_j = u'_i$, and $u_{-\{i,j\}} = u'_{-\{i,j\}}$. Given $k \in N$ and $\tilde{u}_{-k} \in \mathcal{U}^{n-1}$, let $r^*_k(\tilde{u}_{-k}) = \max\{\max_{h \neq k} \min\{\tilde{v}_h, \tilde{b}_h\}, r_k(\tilde{u}_{-k})\}$. By $\max_{k \neq i} \min\{v_k, b_k\} = \max_{k \neq j} \min\{v'_k, b'_k\}$ and upper anonymity of r, $r^*_i(u_{-i}) = r^*_j(u'_{-j})$. If $x_i(u) = x_j(u')$, then by $r^*_i(u_{-i}) = r^*_j(u'_{-j})$, $f_i(u) = f_j(u')$, and so $u_i(f_i(u)) = u_i(f_j(u'))$. Assume $x_i(u) \neq x_j(u')$. Without loss of generality, assume $x_i(u) = 1$ and $x_j(u') = 0$. By $x_i(u) = 1$, $\min\{v_i, b_i\} \geq r^*_i(u_{-i})$, and by $x_j(u') = 0$, $\min\{v'_j, b'_j\} \leq r^*_j(u'_{-j})$. By $\min\{v_i, b_i\} = \min\{v'_j, b'_j\}$ and $r^*_i(u_{-i}) = r^*_j(u'_{-j})$, $\min\{v_i, b_i\} = r^*_i(u_{-i}) = r^*_j(u'_{-j}) = \min\{v'_j, b'_j\}$. By $x_i(u) = 1$ and $r^*_i(u_{-i}) = \min\{v_i, b_i\}$, $f_i(u) = (1, \min\{v_i, b_i\})$. By $x_j(u') = 0$, $f_j(u') = (0,0)$. By $x_j(u') = 0$, $\min\{v'_j, b'_j\} = r^*_j(u'_{-j})$, and a prioritized tie-breaking rule, $v'_j \neq \infty$. By $v_i = v'_j \neq \infty$, $u_i(1, \min\{v_i, b_i\}) = u_i(0,0)$. By $f_i(u) = (1, \min\{v_i, b_i\})$, $f_j(u') = (0,0)$, and $u_i(1, \min\{v_i, b_i\}) = u_i(0,0)$, $u_i(f_i(u)) = u_i(f_j(u'))$. □

References

[1] Adachi, T. (2014), Equity and the Vickrey allocation rule on general preference domains, *Social Choice and Welfare*, 42, 813–830.

[2] Aggarwal, G., S. Muthukrishnan, D. Pál, M. Pál (2009), General auction mechanism for search advertising, In *the 18th International Conference on World Wide Web (WWW'09)*, Madrid, Spain, 241–250.

[3] Ashlagi, I., S. Serizawa (2012), Characterizing Vickrey allocation rule by anonymity, *Social Choice and Welfare*, 38, 53-542.

[4] Ausubel, L. (2004), An efficient dynamic auction for heterogenous commodities. *American Economic Review*, 96, 602–629.

[5] Basu, R., C Mukherjee (2022), Characterization of Vickrey auction with reserve price for multiple objects, *Review of Economic Design*, forthcoming.

[6] Bulow, J., J. Levin, P. Milgrom (2009), Winning play in spectrum auctions. Technical report, *National Bureau of Economic Research.*

[7] Dobzinski, S., R. Lavi, N. Nisan (2012), Multi-unit auctions with budget limits. *Games and Economic Behavior*, 74, 486–503

[8] Dütting, P., M. Henzinger, M. Starnberger (2015), Auctions with heterogeneous items and budget limits. *ACM Transactions on Economics and Computation*, 4, 4:1–4:17.

[9] Dütting, P. M. Henzinger, I. Weber (2015), An Expressive Mechanism for Auctions on the Web. *ACM Transactions on Economics and Computation*, 4, 1:1-1:34.

[10] Fiat, A., S. Leonardi, J. Saia, P. Sankowski (2011), Single valued combinatorial auctions with budgets. In: *the 12th ACM Conference on Electronic Commerce (EC'11)*, San Jose, California, 223–232.

[11] Hirai, H., R. Sato (2023), Polyhedral Clinching Auctions for Indivisible Goods, In: *the 19th International Conference on Web and Internet Economics (WINE 2023)*, Shanghai, China, 366–383.

[12] Holmström, B. (1979), Groves' scheme on restricted domains. *Econometrica*, 47, 1137–1144.

[13] Kazumura, T., D. Mishra, S. Serizawa (2017), Mechanism design without quasilinearity. *ISER Discussion Paper*, No. 1005, Osaka University.

[14] Lavi, R., M. May (2012), A note on the incompatibility of strategy-proofness and Pareto-optimality in quasi-linear settings with public budgets. *Economic Letters*, 115, 100–103.

[15] Le, P. (2017), Mechanisms for combinatorial auctions with budget constraints. *Review of Economic Design*, 21, 1–31.

[16] Le, P. (2018), Pareto optimal budgeted combinatorial auctions. *Theoretical Economics*, 13, 831–868.

[17] Mukherjee, C. (2014), Fair and group strategy-proof good allocation with money, *Social Choice and Welfare*, 42, 289–311

[18] Mackenzie A., Y. Zhou (2022), Tract housing, the core, and pendulum auctions. *Graduate school of Economics Discussion paper series*, No. E-22-005, Kyoto University.

[19] Nisan, N. (2007), Introduction to mechanism design (for computer scientists). In: N. Nisan, T. Rougarden, E. Tardos, V. Vazirani (Eds.), *Algorithmic Game Theory*, Cambridge University Press, New York, USA, 209–241.

[20] Sakai, T. (2008), Second price auctions on general preference domains: two characterizations. *Economic Theory*, 37, 347–356.

[21] Sakai, T. (2013a), An equity characterization of second price auctions when preferences may not be quasilinear, *Review of Economic Design*, 17, 17–26.

[22] Sakai, T. (2013b), Axiomatizations of second price auctions with a reserve price, *International Journal of Economic Theory*, 9, 255-265.

[23] Saitoh, H., S. Serizawa (2008), Vickrey allocation rule with income effect. *Economic Theory*, 35, 391–401.

[24] Shinozaki, H. (2023), Non-obvious manipulability and efficiency in package assignment problems with money for agents with income effects and hard budget constraints. *HIAS Discussion Paper Series*, No. HIAS-E-136, Hitotsubashi University.

[25] Sprumont, Y. (2013), Constrained-optimal strategy-proof assignment: Beyond the Groves mechanisms, *Journal of Economic Theory*, 148, 1102–1121.

[26] Ting, H.F., X. Xiang (2012), Multi-unit auctions with budgets and non-uniform valuations. In: *the 23rd International Symposium on Algorithms and Computation (ISAAC 2012)*, Taipei, Taiwan, 669–678.

[27] Vickrey, W. (1961), Counterspeculation, auctions, and competitive sealed tenders. *Journal of Finance*, 16, 8–37.

[28] Yi, J. (2023), A note on the impossibility of multi-unit auctions with budget-constrained bidders. *Review of Economic Design*, forthcoming.