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An implementation of constrained efficient allocations in hidden information economies

May 24, 2024

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1 Introduction

In an economy with asymmetric information, it is difficult to achieve an efficient allocation as an equilibrium because agents can misreport their information. Therefore, much of the literature focuses on the *constrained efficient allocation*, the second-best allocation among incentive compatible allocations, to explore its attainability within an equilibrium framework.

Prescott and Townsend (1984) is one of the pioneering studies to investigate competitive equilibria in economies with asymmetric information. Their study has inspired numerous researchers to explore various economies with asymmetric information. Rustichini and Siconolfi (2012) consider *economies with observable types*, where agents' types are observable but their action or state are unobservable. In this situation, they demonstrate that some constrained efficient allocation cannot be achieved as a competitive equilibrium, that is, there is no price supporting it.

In this paper, we focus on a simple hidden information economy, which is a special case of economies with observable types as discussed by Rustichini and Siconolfi (2012). Our approach diverges in that we examine the feasibility of implementing a constrained efficient allocation through a Nash equilibrium in a game structure, rather than a competitive equilibrium. We demonstrate that, in almost all cases, achieving the constrained efficient allocation through a subgame perfect equilibrium in a simple game is impossible. This result aligns with the notion that there exists no price that can support a constrained efficient allocation in a general equilibrium framework.

1.1 Related literature

We briefly review the related literature and discuss about our contribution. In the studies about economies with asymmetric information, two main approaches exist for reaching the constrained efficient allocation. The first is the *Walrasian approach*, which seeks to identify prices that support the constrained efficient allocation within a competitive market. The second is the *Nash approach*, aiming to find a Nash equilibrium that implements the constrained efficient allocation within a game structure.

A seminal work employing the Walrasian approach is by Prescott and Townsend (1984), who examine two crucial economies: one with *moral hazard* and the other with *adverse selection*. They demonstrate that the welfare theorems for the constrained efficient allocation hold in an economy with moral hazard, but may fail in one with adverse selection.

Rothschild and Stiglitz (1976) are pioneers in applying the Nash approach. They show that a Cournot-Nash equilibrium may not exist in an economy with adverse selection. Conversely, in an economy with moral hazard, Bennardo and Chiappori (2003) demonstrate that the constrained efficient allocation can be implemented by a Nash equilibrium in a simple game.

Given the limitations of the standard approaches in economies with adverse selection, many researchers have explored additional conditions and solution concepts to achieve the constrained efficient allocation (Walrasian approach: Bisin and Gottardi 2006; Citanna and Siconolfi 2016; Azevedo and Gottlieb 2017; Nash approach: Netzer and Scheuer 2014; Diasakos and Koufopoulos 2018; Dosis 2018, 2019, 2022). This contrasts with the relative success in economies with moral hazard.

However, Jerez (2005) points out a flaw in Prescott and Twonsend's (1986) framework, which imposes incentive compatible constraints solely on consumers, rather than firms. He sees this restriction as a potential conceptual issue, questioning how these constraints are enforced in a decentralized economy. He argues for viewing the incentive compatible constraints as restrictions on the set of allocations that firms can offer to consumers. Thus, investigating the attainability of the constrained efficient allocation in economies with moral hazard attached with this natural restriction is also necessary.

In situations where incentive compatible constraints are placed on firms, Rustichini and Siconolfi (2012) take the Walrasian approach. They demonstrate that competitive equilibrium may not exist and the Second Welfare Theorem may fail in an economy with observable types, including one with moral hazard.

The contribution of our paper is to take the Nash approach in a special case of economies with observable types. We show that for almost all cases, we cannot achieve a constrained efficient allocation as a Nash equilibrium in a simple game. Our impossibility result corresponds to that there is no price that support a constrained efficient allocation.[1]

2 Model

We consider a simple hidden information economy. There exist one agent, $m\,(\geq 2)$ identical firms, and two objects. Let $J = \{1, \ldots, m\}$ be the set of firms, and $L = \{1, 2\}$ be the set of objects. The agent has two states, $g(=$ *good*$)$ and $b(=$ *bad*$)$. Let $S = \{g, b\}$ be the set of states. Each state $s \in S$ occurs with the probability $q_s \in (0, 1)$ such that $q_g + q_b = 1$.

Each firm contracts with the agent before the agent knows her state. A (state-dependent) **allocation** is denoted by a vector $x = (x^1_g, x^1_b, x^2_g, x^2_b) \in \mathbb{R}^4$, where x^l_s is the net amount of the good $l \in L$ when the agent's state is $s \in S$. The agent consumes allocations among the **consumptions set** denoted by $X \subseteq \mathbb{R}^4$ with $0 \in X$.

The firms know the agent's preference, but they cannot observe the state of the agent. The agent has a utility function u over X and the firms have a profit function π over X. We assume that the agent and the firms share the same parameters $(\eta^1_g, \eta^1_b, \eta^2_g, \eta^2_b) \in (0, 1)^4$ such that $\eta^1_g + \eta^2_g = 1$ and $\eta^1_b + \eta^2_b = 1$. The parameter η^l_s is the weight for the object $l \in L$ when the state is $s \in S$. By using these parameters, the agent's utility of $x \in X$ is denoted by

$$\begin{aligned} u(x) &= q_g(\eta^1_g v^1_g(x^1_g) + \eta^2_g v^2_g(x^2_g)) + q_b(\eta^1_b v^1_b(x^1_b) + \eta^2_b v^2_b(x^2_b)), \\ &= q_g\eta^1_g v^1_g(x^1_g) + q_b\eta^1_b v^1_b(x^1_b) + q_g\eta^2_g v^2_g(x^2_g) + q_b\eta^2_b v^2_b(x^2_b), \end{aligned}$$

where $v^l_s : \mathbb{R} \to \mathbb{R}$ is a continuous, strictly concave, strictly increasing function such that $v^l_s(0) = 0$. The firms' profit of $x \in X$ is denoted by

$$\begin{aligned} \pi(x) &= -\{q_g(\eta^1_g x^1_g + \eta^2_g x^2_g) + q_b(\eta^1_b x^1_b + \eta^2_b x^2_b)\}, \\ &= -(q_g\eta^1_g x^1_g + q_b\eta^1_b x^1_b + q_g\eta^2_g x^2_g + q_b\eta^2_b x^2_b). \end{aligned}$$

[1]See Section 4 for more detailed discussion.

One of important economies in our model is an insurance economy. We consider a car insurance. Assume $\eta_g^1 < \eta_b^1$ (or equivalently $\eta_g^2 > \eta_b^2$). Here, object 1 represents the amount of money the agent can obtain when he/she is involved in a traffic accident, while object 2 denotes the amount when he/she is not. In this context, η_s^1 represents the probability of an agent getting into a traffic accident when their state is s. Additionally, $\eta_g^1 < \eta_b^1$ indicates that if the agent's state is good, he/she can decrease the probability of being involved in a traffic accident. Thus this situation reflects a car insurance economy.

2.1 Incentive compatibility and constrained efficiency

Let $\phi : \{g, b\} \to \{g, b\}$ be a manipulation which the agent takes, and Φ be the set of all manipulations. Given an allocation $x \in X$ and a manipulation $\phi \in \Phi$, we denote a manipulated utility by

$$u_\phi(x) = q_g\eta_g^1 v_g^1(x_{\phi(g)}^1) + q_b\eta_b^1 v_b^1(x_{\phi(b)}^1) + q_g\eta_g^2 v_g^2(x_{\phi(g)}^2) + q_b\eta_b^2 v_b^2(x_{\phi(b)}^2).$$

Similarly, denote a manipulated profit by

$$\pi_\phi(x) = q_g\eta_g^1 x_{\phi(g)}^1 + q_b\eta_b^1 x_{\phi(b)}^1 + q_g\eta_g^2 x_{\phi(g)}^2 + q_b\eta_b^2 x_{\phi(b)}^2.$$

Let $\phi_I \in \Phi$ be the truth-telling manipulation, that is, $\phi_I(g) = g$ and $\phi_I(b) = b$. Let $\phi_G \in \Phi$ be a manipulation such that she always reports the *good* state, that is, $\phi_G(g) = \phi_G(b) = g$. We similarly define $\phi_B \in \Phi$ by $\phi_B(g) = \phi_B(b) = b$. Let ϕ_E be a manipulation such that she exchanges her states to report, that is, $\phi_E(g) = b$ and $\phi_E(b) = g$. We can rewrite the set of manipulations by $\Phi = \{\phi_I, \phi_G, \phi_B, \phi_E\}$.

An allocation $x \in X$ is **feasible** if $q_g x_g^1 + q_b x_b^1 \leq 0$ and $q_g x_g^2 + q_b x_b^2 \leq 0$. Let $X^F \subseteq X$ be the set of all feasible allocations. An allocation $x \in X$ is **incentive compatible** if for each $\phi \in \Phi$, $u(x) \geq u_\phi(x)$. Let $X^{IC} \subseteq X$ be the set of incentive compatible allocations. We denote the set of all feasible and incentive compatible allocations by $X^{FIC} = X^F \cap X^{IC}$. Note that, since 0 is feasible and incentive compatible, X^{FIC} is non-empty.

An allocation $x \in X$ is **constrained efficient** if (i) $x \in X^{FIC}$, and (ii) there is no $x' \in X^{FIC}$ such that $u(x') > u(x)$. Let $X^{CE} \subseteq X$ be the set of constrained efficient allocations.

2.2 Subgame perfect implementability

We consider the following game form:

Stage 1: Each firm $j \in J$ offers a set of allocations $D_j \subseteq X$.

Stage 2: The agent chooses an allocation $x \in \bigcup_{j \in J} D_j$ and a manipulation $\phi \in \Phi$.

Let $\mathcal{D} = 2^X$ be the firms' strategy space. Let $f : \mathcal{D} \to X \times \Phi$ be a strategy of the agent, and $\mathcal{F}$ be the set of all the agent's strategies. Given $D \in \mathcal{D}$ and $f \in \mathcal{F}$, let $x^f(D)$ and $\phi^f(D)$ be an allocation and a manipulation obtained by the strategies.

Given $D \in \mathcal{D}$, let $1_D : X \to \{0,1\}$ be a indicator function such that for each $x \in X$, $1_D(x) = 1$ if $x \in D$ and $1_D(x) = 0$ if $x \notin D$. A profile of strategies $((D_j)_{j\in J}, f)$ is a **subgame perfect equilibrium** if

(i) for each $D \in \mathcal{D}$,

$$(x^f(D), \phi^f(D)) \in \underset{(x,\phi)\in D\times\Phi}{\arg\max}\, u_\phi(x),$$

(ii) for each $j \in J$ and $D'_j \in \mathcal{D}$,

$$\frac{1_{D_j}(x^f(D))}{\sum_{k\in J} 1_{D_k}(x^f(D))}\,\pi_{\phi^f(D)}(x^f(D)) \geq \frac{1_{D'_j}(x^f(D'))}{1_{D'_j}(x^f(D')) + \sum_{k\neq j} 1_{D_k}(x^f(D'))}\,\pi_{\phi^f(D')}(x^f(D')),$$

where $D = \cup_{k\in J} D_k$ and $D' = D'_j \cup (\bigcup_{k\neq j} D_k)$. The first condition is about the agent's optimality, and the second condition is about the firms' optimality.

An allocation $x \in X$ is **subgame perfect implementable** if there exists a subgame perfect equilibrium $((D_j)_{j\in J}, f) \in \mathcal{D}^m \times \mathcal{F}$ such that $(x^f(D), \phi^f(D)) = (x, \phi_I)$, where $D = \bigcup_{j\in J} D_j$.

3 Result

Our main result states that if the parameters are not symmetric, then any non-zero constrained efficient allocation is not subgame perfect implementable.

Theorem 1. Assume that $\eta_g^1 \neq \eta_b^1$ or $\eta_g^2 \neq \eta_b^2$. Then, for each $x \in X^{CE} \setminus \{0\}$, x is not subgame perfect implementable.

In the following, we undertake the proof of Theorem 1. Initially, we present some preliminary results. Subsequently, we analyze certain properties of the constrained efficient allocation. Finally, we proceed to establish our main theorem.

3.1 Preliminaries

The first result shows the necessary and sufficient condition for incentive compatibility.

Lemma 1. For each $x \in X$, $x \in X^{IC}$ if and only if $u(x) \geq u_{\phi_G}(x)$ and $u(x) \geq u_{\phi_B}(x)$.

Proof. Let $x \in X$. If $x \in X^{IC}$, then $u(x) \geq u_{\phi_G}(x)$ and $u(x) \geq u_{\phi_B}(x)$ are obvious. Thus, assume $u(x) \geq u_{\phi_G}(x)$ and $u(x) \geq u_{\phi_B}(x)$, and show $u(x) \geq u_{\phi_E}(x)$. By definition,

$$\begin{aligned}
& u_{\phi_G}(x) + u_{\phi_B}(x) \\
= & \begin{array}{l} (q_g\eta_g^1 v_g^1(x_g^1) + q_b\eta_b^1 v_b^1(x_g^1) + q_g\eta_g^2 v_g^2(x_g^2) + q_b\eta_b^2 v_b^2(x_g^2)) \\ + \,(q_g\eta_g^1 v_g^1(x_b^1) + q_b\eta_b^1 v_b^1(x_b^1) + q_g\eta_g^2 v_g^2(x_b^2) + q_b\eta_b^2 v_b^2(x_b^2)) \end{array} \\
= & u(x) + u_{\phi_E}(x).
\end{aligned}$$

Hence, by $u(x) \geq u_{\phi_G}(x)$ and $u(x) \geq u_{\phi_B}(x)$,

$$u(x) + u(x) \geq u_{\phi_G}(x) + u_{\phi_B}(x) = u(x) + u_{\phi_E}(x),$$

which implies $u(x) \geq u_{\phi_E}(x)$. Therefore, $x \in X^{CE}$. □

Given $x \in X$, by a simple calculation,

$$\begin{aligned} & u(x) \geq u_{\phi_G}(x) \\ \Leftrightarrow \; & \quad q_g\eta_g^1 v_g^1(x_g^1) + q_b\eta_b^1 v_b^1(x_b^1) + q_g\eta_g^2 v_g^2(x_g^2) + q_b\eta_b^2 v_b^2(x_b^2) \\ & \geq q_g\eta_g^1 v_g^1(x_g^1) + q_b\eta_b^1 v_b^1(x_g^1) + q_g\eta_g^2 v_g^2(x_g^2) + q_b\eta_b^2 v_b^2(x_g^2) \\ \Leftrightarrow \; & q_b\eta_b^1 v_b^1(x_b^1) + q_b\eta_b^2 v_b^2(x_b^2) \geq q_b\eta_b^1 v_b^1(x_g^1) + q_b\eta_b^2 v_b^2(x_g^2) \\ \Leftrightarrow \; & \eta_b^1(v_b^1(x_b^1) - v_b^1(x_g^1)) \geq \eta_b^2(v_b^2(x_g^2) - v_b^2(x_b^2)). \end{aligned}$$

In the same way, we can show that $u(x) \geq u_{\phi_B}(x)$ if and only if $\eta_g^2(v_g^2(x_g^2) - v_g^2(x_b^2)) \geq \eta_g^1(v_g^1(x_b^1) - v_g^1(x_g^1))$. Hence, we get the following result from Lemma 1.

Corollary 1. For each $x \in X$, $x \in X^{IC}$ if and only if

$$\begin{aligned} &\text{(i)} \quad \eta_b^1(v_b^1(x_b^1) - v_b^1(x_g^1)) \geq \eta_b^2(v_b^2(x_g^2) - v_b^2(x_b^2)), \\ &\text{(ii)} \quad \eta_g^2(v_g^2(x_g^2) - v_g^2(x_b^2)) \geq \eta_g^1(v_g^1(x_b^1) - v_g^1(x_g^1)). \end{aligned}$$

The next result is about the relation between the amounts of the object 1 and object 2.

Lemma 2. For each $x \in X^{IC}$, $x_b^1 \geq x_g^1$ if and only if $x_g^2 \geq x_b^2$.

Proof. Let $x \in X^{IC}$. If $x_b^1 \geq x_g^1$, then since v_g^1 is increasing, $v_g^1(x_b^1) - v_g^1(x_g^1) \geq 0$, and so by Corollary 1 (ii),

$$\eta_g^2(v_g^2(x_g^2) - v_g^2(x_b^2)) \geq \eta_g^1(v_g^1(x_b^1) - v_g^1(x_g^1)) \geq 0.$$

Since v_g^2 is increasing, by the above inequality, $x_g^2 \geq x_b^2$. Similarly, if $x_g^2 \geq x_b^2$, then by Corollary 1 (i), $x_b^1 \geq x_g^1$. □

3.2 Properties

The first result states that for any non-zero constrained efficient allocation, at least one incentive compatible constraint holds with strict inequality.

Proposition 1. For each $x \in X^{CE} \setminus \{0\}$, $u(x) > u_{\phi_G}(x)$ or $u(x) > u_{\phi_B}(x)$.

Proof. Let $x \in X^{CE} \setminus \{0\}$. Suppose that $u(x) = u_{\phi_G}(x)$ and $u(x) = u_{\phi_B}(x)$. We show $u(x) < 0 = u(0)$. This is a contradiction to constrained efficiency because 0 is feasible and incentive compatible.

Without loss of generality, assume $x_b^1 \geq x_g^1$. The converse case can be demonstrated similarly. If $x_b^1 = x_g^1$, then by Lemma 2, $x_g^2 = x_b^2$, and so by feasibility, $x_g^1 = x_b^1 = x_g^2 = x_b^2 = 0$. However, this contradicts $x \neq 0$. Hence, $x_b^1 > x_g^1$. By Lemma 2, $x_g^2 > x_b^2$.

We show that $\eta_g^1 v_g^1(x_g^1)+\eta_g^2 v_g^2(x_g^2)<0$ and $\eta_b^1 v_b^1(x_b^1)+\eta_b^2 v_b^2(x_b^2)<0$. These two inequalities imply $u(x)<0$. Since we can prove each inequality in the same way, we only show $\eta_g^1 v_g^1(x_g^1)+\eta_g^2 v_g^2(x_g^2)<0$. By $x_g^1<x_b^1$ and feasibility, $q_g x_g^1+q_b x_g^1<q_g x_g^1+q_b x_b^1\leq 0$, and so $x_g^1<0$. Similarly, by $x_b^2<x_g^2$ and feasibility, $x_b^2<0$.

CASE 1: $x_g^1<x_b^1\leq 0$.

By Corollary 1 (ii),

$$\eta_g^2 v_g^2(x_g^2) \underset{0>x_b^2}{<} \eta_g^2(v_g^2(x_g^2)-v_g^2(x_b^2)) \underset{u(x)=u_{\phi_B}(x)}{=} \eta_g^1(v_g^1(x_b^1)-v_g^1(x_g^1)) \underset{x_b^1\leq 0}{\leq} -\eta_g^1 v_g^1(x_g^1),$$

and so $\eta_g^1 v_g^1(x_g^1)+\eta_g^2 v_g^2(x_g^2)<0$.

CASE 2: $x_g^1<0<x_b^1$.

If $x_g^2\leq 0$, then by $x_g^1<0$, $\eta_g^1 v_g^1(x_g^1)+\eta_g^2 v_g^2(x_g^2)<0$. Hence, we assume $x_g^2>0$. By $x_b^2<0$, $x_b^2<0<x_g^2$. By strict concavity,

$$\frac{x_b^1}{x_b^1-x_g^1}\cdot v_g^1(x_g^1)+\frac{-x_g^1}{x_b^1-x_g^1}\cdot v_g^1(x_b^1)<v_g^1\left(\frac{x_b^1}{x_b^1-x_g^1}\cdot x_g^1+\frac{-x_g^1}{x_b^1-x_g^1}\cdot x_b^1\right)=v_g^1(0)=0. \tag{1}$$

By (1),

$$\begin{aligned}
&-v_g^1(x_g^1)+\left(\frac{x_b^1}{x_b^1-x_g^1}\cdot v_g^1(x_g^1)+\frac{-x_g^1}{x_b^1-x_g^1}\cdot v_g^1(x_b^1)\right)<-v_g^1(x_g^1)+0\\
\Leftrightarrow\ &\frac{x_g^1}{x_b^1-x_g^1}\cdot v_g^1(x_g^1)+\frac{-x_g^1}{x_b^1-x_g^1}\cdot v_g^1(x_b^1)<-v_g^1(x_g^1)\\
\Leftrightarrow\ &\frac{-x_g^1}{x_b^1-x_g^1}\cdot (v_g^1(x_b^1)-v_g^1(x_g^1))<-v_g^1(x_g^1)\\
\Leftrightarrow\ &\frac{-x_g^1}{x_b^1-x_g^1}\cdot \eta_g^1(v_g^1(x_b^1)-v_g^1(x_g^1))<-\eta_g^1 v_g^1(x_g^1)
\end{aligned} \tag{2}$$

In the same way, by $x_b^2<0<x_g^2$ and strict concavity, we can show

$$\eta_g^2 v_g^2(x_g^2)<\frac{x_g^2}{x_g^2-x_b^2}\cdot\eta_g^2(v_g^2(x_g^2)-v_g^2(x_b^2)). \tag{3}$$

By feasibility,

$$\frac{x_g^2}{x_g^2-x_b^2} \underset{q_g x_g^2+q_b x_b^2\leq 0}{\leq} \frac{q_b x_g^2-q_b x_b^2}{x_g^2-x_b^2}=q_b=\frac{q_b x_b^1-q_b x_g^1}{x_b^1-x_g^1} \underset{q_g x_g^1+q_b x_b^1\leq 0}{\leq} \frac{-x_g^1}{x_b^1-x_g^1}. \tag{4}$$

By $u(x)=u_{\phi_B}(x)$ and Corollary 1 (ii),

$$\eta_g^2(v_g^2(x_g^2)-v_g^2(x_b^2))=\eta_g^1(v_g^1(x_b^1)-v_g^1(x_g^1)). \tag{5}$$

Hence,

$$\eta_g^2 v_g^2(x_g^2) \underset{(3)}{<} \frac{x_g^2}{x_g^2 - x_b^2} \cdot \eta_g^2(v_g^2(x_g^2) - v_g^2(x_b^2)) \underset{(4,5)}{\leq} \frac{-x_g^1}{x_b^1 - x_g^1} \cdot \eta_g^1(v_g^1(x_b^1) - v_g^1(x_g^1)) \underset{(2)}{<} -\eta_g^1 v_g^1(x_g^1),$$

and so $\eta_g^1 v_g^1(x_g^1) + \eta_g^2 v_g^2(x_g^2) < 0$.

Similarly, we can show $\eta_b^1 v_b^1(x_b^1) + \eta_b^2 v_b^2(x_b^2) < 0$, and so $u(x) < 0$. □

The next result states that the feasibility constraints are binding for any constrained efficient allocation.

Proposition 2. For each $x \in X^{CE}$, $q_g x_g^1 + q_b x_b^1 = 0$ and $q_g x_g^2 + q_b x_b^2 = 0$.

Proof. Let $x \in X^{CE}$. If $x = 0$, then the result is obvious, and so we assume $x \in X^{CE} \setminus \{0\}$. By $x \in X^{CE} \setminus \{0\}$ and Proposition 1, $u(x) > u_{\phi_G}(x)$ or $u(x) > u_{\phi_B}(x)$. We only show $q_g x_g^1 + q_b x_b^1 = 0$ since we can prove $q_g x_g^2 + q_b x_b^2 = 0$ in the same way. To show the result, suppose $q_g x_g^1 + q_b x_b^1 < 0$.

Without loss of generality, assume $u(x) > u_{\phi_G}(x)$. We can derive a contradiction in the same way for the case $u(x) > u_{\phi_B}(x)$. By $u(x) > u_{\phi_G}(x)$ and Corollary 1 (i), $\eta_b^1(v_b^1(x_b^1) - v_b^1(x_g^1)) > \eta_b^2(v_b^2(x_g^2) - v_b^2(x_b^2))$. Let $\varepsilon > 0$ be such that $q_g(x_g^1 + \varepsilon) + q_b x_b^1 < 0$ and

$$\eta_b^1(v_b^1(x_b^1) - v_b^1(x_g^1)) > \eta_b^1(v_b^1(x_b^1) - v_b^1(x_g^1 + \varepsilon)) > \eta_b^2(v_b^2(x_g^2) - v_b^2(x_b^2)). \tag{6}$$

By Corollary 1 (ii),

$$\eta_g^2(v_g^2(x_g^2) - v_g^2(x_b^2)) \geq \eta_g^1(v_g^1(x_b^1) - v_g^1(x_g^1)) > \eta_g^1(v_g^1(x_b^1) - v_g^1(x_g^1 + \varepsilon)) \tag{7}$$

Hence, by (6), (7), and Corollary 1, $(x_g^1 + \varepsilon, x_b^1, x_g^2, x_b^2)$ is incentive compatible. However, since $u(x_g^1 + \varepsilon, x_b^1, x_g^2, x_b^2) > u(x)$ and $(x_g^1 + \varepsilon, x_b^1, x_g^2, x_b^2)$ is feasible, this contradicts constrained efficiency. □

3.3 Proof of Theorem 1

Finally, we show Theorem 1.

Proof of Theorem 1. Without loss of generality, assume $\eta_g^1 > \eta_b^1$. By $\eta_g^1 + \eta_g^2 = 1$ and $\eta_b^1 + \eta_b^2 = 1$, $\eta_g^2 < \eta_b^2$. Let $x \in X^{CE} \setminus \{0\}$. If $x_g^1 = x_b^1$, then by Lemma 2 and Proposition 2, $x_g^1 = x_b^1 = x_g^2 = x_b^2 = 0$. However, this contradicts $x \in X^{CE} \setminus \{0\}$. Hence, $x_g^1 \neq x_b^1$.

Case 1: $x_g^1 > x_b^1$.

By Lemma 2, $x_g^2 < x_b^2$. By Proposition 2, $x_g^1 > 0 > x_b^1$ and $x_g^2 < 0 < x_b^2$. By $\eta_g^1 > \eta_b^1$, $x_g^1 > 0 > x_b^1$ and Proposition 2,

$$q_g \eta_g^1 x_g^1 + q_b \eta_b^1 x_b^1 > q_g \eta_b^1 x_g^1 + q_b \eta_b^1 x_b^1 = 0. \tag{8}$$

By $\eta_g^2 < \eta_b^2$, $x_g^2 < 0 < x_b^2$ and Proposition 2,

$$q_g \eta_g^2 x_g^2 + q_b \eta_b^2 x_b^2 > q_g \eta_g^2 x_g^2 + q_b \eta_g^2 x_b^2 = 0. \tag{9}$$

By (8) and (9), $\pi(x) < 0$. If a firm j offers $\{0\}$, then j's profit becomes zero. Hence, x is not subgame perfect implementable.

CASE 2: $x_g^1 < x_b^1$.

By Lemma 2, $x_g^2 > x_b^2$. By Proposition 2, $x_g^1 < 0 < x_b^1$ and $x_g^2 > 0 > x_b^2$. By $\eta_g^1 > \eta_b^1$, $x_g^1 < 0 < x_b^1$ and Proposition 2,

$$q_g\eta_g^1 x_g^1 + q_b\eta_b^1 x_b^1 < q_g\eta_g^1 x_g^1 + q_b\eta_g^1 x_b^1 = 0. \tag{10}$$

By $\eta_g^2 < \eta_b^2$, $x_g^2 > 0 > x_b^2$ and Proposition 2,

$$q_g\eta_g^2 x_g^2 + q_b\eta_b^2 x_b^2 < q_g\eta_b^2 x_g^2 + q_b\eta_b^2 x_b^2 = 0. \tag{11}$$

By (10) and (11), $\pi(x) > 0$. By $x \in X^{CE} \setminus \{0\}$ and Proposition 1, $u(x) > u_{\phi_G}(x)$ or $u(x) > u_{\phi_B}(x)$. Without loss of generality, assume $u(x) > u_{\phi_G}(x)$. By Corollary 1 (i), $\eta_b^1(v_b^1(x_b^1) - v_b^1(x_g^1)) > \eta_b^2(v_b^2(x_g^2) - v_b^2(x_b^2))$. Let $\varepsilon > 0$ be such that $\pi(x) > \pi(x_g^1+\varepsilon, x_b^1, x_g^2, x_b^2) > \frac{1}{m}\pi(x)$ and

$$\eta_b^1(v_b^1(x_b^1) - v_b^1(x_g^1)) > \eta_b^1(v_b^1(x_b^1) - v_b^1(x_g^1+\varepsilon)) > \eta_b^2(v_b^2(x_g^2) - v_b^2(x_b^2)). \tag{12}$$

By Corollary 1 (ii),

$$\eta_g^2(v_g^2(x_g^2) - v_g^2(x_b^2)) \geq \eta_g^1(v_g^1(x_b^1) - v_g^1(x_g^1)) > \eta_g^1(v_g^1(x_b^1) - v_g^1(x_g^1+\varepsilon)). \tag{13}$$

Let $y = (x_g^1+\varepsilon, x_b^1, x_g^2, x_b^2)$. By (12), (13) and Corollary 1, $u(y) > \min\{u_{\phi_G}(y), u_{\phi_B}(y), u_{\phi_E}(y)\}$. If firm j offers $\{y\}$, then by $u(y) > u(x)$ and $u(y) > \min\{u_{\phi_G}(y), u_{\phi_B}(y), u_{\phi_E}(x)\}$, the agent chooses y and takes ϕ_I. Hence, by $\pi(y) > \frac{1}{m}(x)$, x is not subgame perfect implementable. $\square$

4 Discussion

In this section, we explore the relationship between our result and Rustichini and Siconolfi's (2012) result, hereafter referred to as H & S. To accomplish this, we begin by defining a competitive equilibrium. Let $\sigma \in \Delta(X)$ be a lottery on X, and let $p \in \mathbb{R}^X$ be a price vector over X. A pair (σ, p) is a **competitive equilibrium** if (i) $\sigma \in \arg\max_{\sigma' \in \Delta(X)} \sum_{x \in X} u(x)\sigma'(x)$ subject to $\sum_{x \in X} p(x)\sigma'(x) \leq 0$, (ii) $\sigma \in \arg\max_{\sigma' \in \Delta(X^{FIC})} p(x)\sigma'(x)$, and (iii) $\sigma \in \Delta(X^F)$. The first condition is about utility maximization, the second is about profit maximization, and the third is feasibility condition.

Given $x \in X$, let $\delta_x \in \Delta(X)$ be the degenerate lottery such that $\delta_x(x) = 1$ and $\delta_x(x') = 0$ for $x' \neq x$. To be consistent with H & S's model, given a equilibrium price p, the profit function must satisfy $\pi(x) = \sum_{x' \in X} p(x')\delta_x(x') = p(x)$ for all $x \in X$.

R & H establish the existence of competitive equilibria through two steps. First, they identify a price vector that supports the constrained efficient allocation, i.e., the allocation maximizes agent's utility at the price. Next, they verify whether the constrained efficient allocation maximizes the firm's profit under the obtained price.

R & H show the necessary and sufficient condition for a price vector to support the constrained efficient allocation, termed the *price supportability condition*. If the constrained efficient allocation is degenerate for x, then the price supportability condition says that $p(x) = 0$ and for each $x' \neq x$, $p(x') \geq 0$. As we have shown in Theorem 1, if parameters η_s^l are asymmetric, then it follows that $\pi(x) \neq 0$, and hence $p(x) \neq 0$. Thus, the result of Theorem 1 is attributed to the absence of price supportability.

Finally, we illustrate that even when the price supportability condition is satisfied and the profit is maximized, indicating the existence of a competitive equilibrium, the constrained efficient allocation is not subgame perfect implemtnable. This distinction arises from the fact that firms only offer feasible allocations in R & S's model, whereas they can offer non-feasible allocations in ours.

Example 1. Let $q_g = q_b = 0.5$ and $\eta_g^1 = \eta_b^1 = \eta_g^2 = \eta_b^2 = 0.5$. Let u be such that for each $x \in X$,

$$u(x) = x_g^1(2 - x_g^1) + 3x_b^1(2 - x_b^1) + 6x_g^2(2 - x_g^2) + 2x_b^2(2 - x_b^2).$$

Then, $x^* = (-0.5, 0.5, 0.5, -0.5)$ is the unique constrained efficient allocation. By $\pi(x^*) = 0$, the price supportability condition is satisfied. Moreover, for each $x \in X^{FIC}$, feasibility conditions, $q_g x_g^1 + q_b x_b^1 \leq 0$ and $q_g x_g^2 + q_b x_b^2 \leq 0$, and symmetric parameters, $\eta_g^1 = \eta_b^1$ and $\eta_g^2 = \eta_b^2$, imply the nonpositive profit $\pi(x) \leq 0$. Thus, x^* maximizes the profit over X^{FIC}, and so (x^*, p^*) is a competitive equilibrium, where $p^*(x) = \pi(x)$ for $x \in X$.

However, x^* is not subgame perfect implementable. To see this, let $\tilde{x} = (-0.64, 0.5, 0.6, -0.5)$. Then,

$$\begin{aligned} u(\tilde{x}) &= -1.6896 + 2.25 + 5.04 - 2.5 = 3.1004, \\ u_{\phi_G}(\tilde{x}) &= -1.6896 - 5.0688 + 5.04 + 1.68 = -0.0384, \\ u_{\phi_B}(\tilde{x}) &= 0.75 + 2.25 - 7.5 - 2.5 = -7, \\ \pi(\tilde{x}) &= -0.25(-0.64 + 0.5 + 0.6 - 0.5) = 0.01. \end{aligned}$$

Hence, by $u(\tilde{x}) = 3.1004 > 3 = u(x^*)$, $u(\tilde{x}) > \min\{u_{\phi_G}(\tilde{x}), u_{\phi_B}(\tilde{x}), u_{\phi_E}(\tilde{x})\}$, and $\pi(\tilde{x}) = 0.01 > 0 = \pi(x^*)$, x^* is not subgame perfect implementable. This is because we assume that the firms can offer non-feasible allocations in the game.

5 Conclusion

In this study, we examine a simple hidden information economy. We demonstrate that when the parameters of a utility function exhibit asymmetry, any non-zero constrained efficient allocation is not subgame perfect implementable. We leave the question of subgame perfect implementability open in the case of symmetric parameters.

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