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Advertisement versus Motivation in Competitive Search Equilibrium

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【要約】We analyze equilibrium wage contracts in a competitive search model where a firm motivates workers to invest in a match-specific skill. If skill is not critical for production, the contract is first best. If critical, the contract coincides with an efficiency wage contract and cannot attain even second best. Unlike standard efficiency wage models, the wage plays a dual role, advertisement and motivation, which induces a novel source of inefficiency: the competition to attract workers forces a wage to be chosen that increases the ex ante utility of workers at the expense of ex post utility.

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1 Introduction

Firm-specific human capital can improve firm productivity, create rents, and increase the retention of workers. Consequently, there is little doubt that it is important for firms to motivate workers to invest in firm-specific human capital. For the most part, contract theories have proven useful in informing us how to design a wage contract to encourage this type of investment.

Although contract theories are typically abstracted from outside markets, wage contracts are obviously also important in determining the prices for labor. Market competition must then influence these contracts. That is, when firms write contracts, they must consider not only the incentives of employed workers, but also the competition to attract potential applicants. How does this competitive pressure influence the wage contract offered in the market? Further, what are the welfare consequences of competition when a firm must motivate its workers?

This paper analyzes an equilibrium wage contract when a firm must motivate workers to invest in match-specific human capital in a competitive search model. As emphasized in Albrecht (2011), the competitive search model is a particularly useful benchmark because it is known to attain the constrained optimal allocation[1]. Any deviation from the standard competitive search model then allows us to identify the source of inefficiency[2]. This paper investigates a possible reason for inefficiency when a firm must motivate workers to invest in match-specific human capital in a frictional labor market.

We summarize our main results as follows. If match-specific human capital is not critical for production, we show that market competition does not interfere with the incentive contract. In order to motivate workers, the differences between the wages for successful workers and those for failed workers must be large. Conversely,

[1]Several studies (e.g., Moen (1997), Shimer (1996), Acemoglu and Shimer (1999b), Shi (2001), Shimer (2005) and Menzio and Shi (2011)) establish that the competitive search equilibrium attains the constrained optimal under several different environments.

[2]For example, Albrecht, Gautier and Vroman (2006), Guerrieri (2008), Galenianos and Kircher (2009), Galenianos, Kircher and Gabor (2011), and Delacroix and Shi (2013) analyze the possible source of inefficiencies using competitive search models.

the average level of wage payment is important for the competition to attract risk-neutral workers. Hence, it is possible to design a wage contract that simultaneously incorporates both incentives and advertisement. We then show that the equilibrium wage contract can induce optimal effort and optimal labor market tightness under the constraint of search friction, which we refer to as the first best.

However, if match-specific human capital is critical for production, the result significantly changes. Because the failure to obtain match-specific human capital lowers the rent from their long-term relationship, it diminishes the reasons for the firm and the worker to maintain their relationship. Because workers can simply walk away from the firm, the value of unemployed workers is the lowest bound of a punishment that a firm can impose on workers when they fail to obtain match-specific human capital. We show that when match-specific human capital is critical, this constraint for the lowest bound is more likely to bind, and the value of the failed workers then equals that of unemployed workers. As a result, the equilibrium contract is essentially a version of an efficiency wage contract.

However, unlike a standard efficiency wage model, the wage contract in our model must play a dual role; namely, advertisement and motivation. Note that when the value of the failed workers is determined by that of the unemployed workers, the wage payment for successful workers must influence both the average level of the wage payments and the difference between the wage payments for successful workers and failed workers. We show that a worker's incentive compatibility condition and a firm's zero-profit condition bring about a trade-off between the workers' probability of finding a job and their investment effort after locating a partner. That is, a firm must pay a higher wage to induce greater effort. However, because the promise of a higher wage reduces firms' profits, only a smaller number of firms can enter the market, which reduces labor market tightness and, therefore, the probability of workers finding a job.

Because of this trade-off, it is impossible to increase equilibrium effort and a job-finding probability at the same time. Hence, both the equilibrium effort level and a job-finding probability are lower than the first best. We then question whether the competitive search equilibrium can attain the second best, which maximizes the social welfare subject to a worker's incentive compatibility condition and a firm's zero-profit condition. For this purpose, we examine whether a tax on labor market tightness can increase welfare under the competitive search equilibrium. Because the tax benefits can be used to finance unemployment benefits, the only role of the labor market tightness tax is to direct unemployed workers to select a labor market with less labor market tightness. If the competitive search equilibrium attains the second best, the optimal tax rate should be 0. However, we find that a slight increase in the tax from 0 can improve welfare.

We identify two potential sources of inefficiency: *a misdirected effect* and *a negative externality effect*. As Acemoglu and Shimer (1999a) and Guerrieri (2008) argue, only wage contracts that maximize the utility of unemployed workers survive in a competitive search equilibrium. However, the social planner needs to care not only about the utility of unemployed workers, but also about that of employed workers. We show that maximizing the utility of unemployed workers also maximizes social welfare, but only if workers' ex post effort is optimal. Therefore, if workers' ex post effort is less than optimal, it is possible to improve welfare by providing additional rent to successful workers to induce effort. However, because this reduces the job-finding probability, it lowers the utility of unemployed workers. Therefore, this type of wage contract cannot survive in a competitive economy. Thus, when the wage must play an advertisement and an incentive role at the same time, the competition to attract unemployed workers forces firms to offer that wage that improves the ex ante utility of workers at the expense of ex post utility. We refer to this as *misdirected effect*.

The competition to attract workers also has another effect. The competition to attract workers increases the utility of unemployed workers, which makes it costly for other firms to provide workers with appropriate incentives. Hence, when there is a trade-off between high effort and significant labor market tightness, the competition to attract workers not only misdirects an individual firm's decision, but also interferes with other firms' decisions. We call this the *negative externality effect*.

Although both the misdirected effect and the negative externality effect can influence welfare, the marginal benefits from mitigating the negative externality effect is almost 0 when the labor market tightness tax rate is close to 0. Because labor market tightness is chosen to maximize the utility of unemployed workers under a competitive search equilibrium, the marginal changes in labor market tightness cannot lower the utility of unemployed workers very much and, therefore, cannot counter the negative externality effect. However, because the trade-off between effort and labor market tightness exists, even when the tax rate is 0, small changes in labor market tightness can lower the misdirected effect. Because of the misdirected effect, we show that a slight increase in labor market tightness tax under the competitive search equilibrium can improve welfare.

Note that the notion of misdirected effect is entirely novel. On the one hand, because the standard efficiency wage model focuses on the incentive role of wage payment, it ignores the advertisement role of the wage contract. Hence, although the negative externality effect exists in a standard efficiency wage model, the misdirected effect does not. On the other hand, because the standard competitive search model does not include any interaction between the incentive scheme and market competition, the competition to attract unemployed workers does not distort the incentive to increase ex post surplus. Hence, there is neither a misdirected effect nor a negative externality effect. To our knowledge, this is the first study to point out the importance of this misdirected effect.

We also show that if a trade-off between high effort and significant labor market tightness is severe, there is bargaining power under which a search model with wage bargaining can attain a larger social surplus than the competitive search model. This is surprising because it is well known that welfare under a search model with wage bargaining cannot be greater than that in a standard competitive search model. This result indicates that the misdirection of the competitive wage setting may be quite large when the wage must play not only the role of advertisement, but also that of an incentive.

Finally, we show that if an up-front fee is acceptable, the equilibrium contract can always attain the first best. In particular, if up-front fees are acceptable, workers are willing to pay these fees if the firm promises a sufficiently high wage when they succeed in investing in match-specific human capital. This is feasible because if this rearrangement induces optimal effort, it can generate greater surplus. In other words, we can consider up-front fees as a transfer mechanism from ex ante surplus to ex post surplus to motivate workers to make an effort. If this transfer mechanism exists, the wage contract that maximizes unemployed workers also maximizes social welfare. This suggests that the lack of a transfer mechanism because of limited liability is necessary to derive our result.

Prescott and Townsend (1984a, 1984b) investigate an optimal contract under private information in a competitive market. In particular, they examine whether a decentralized economy can implement the constrained optimal allocation. Recently, many studies introduce market friction and examine the private information in a competitive search framework (e.g., Guerrieri (2008), Guerrieri, Shimer and Wright (2010), and Delacroix and Shi (2013)). In particular, and similar to our analysis, Guerrieri (2008) points out the importance of the negative externality through the endogenous value of unemployed workers. She finds that the market allocation cannot attain the constrained optimal because of the negative externality effect. However, unlike her model, we emphasize yet another source of inefficiency, namely, the misdirected effect, and show that only a slight increase in the labor market tightness tax under the competitive search equilibrium can improve welfare because of this misdirected effect.

The literature on efficiency wage models emphasizes a high wage as a device to motivate workers (e.g., Shapiro and Stiglitz (1984) and MacLeod and Malcomson (1998)). Although most of the existing literature considers the situation where output cannot be contractible, Moen and Rosén (2006) challenge this view and construct an efficiency wage model where output is contractible. However, when firms post the contract, they do not take into account the trade-off between labor market tightness and the wage contract. Therefore, there is no trade-off between advertisement and incentive, and the posted wage contract is constrained efficient.

Later, Moen and Rosén (2011) explicitly consider the dual roles of the wage contract as motivation and advertisement and analyze the equilibrium contract in a competitive search model. However, because they pay more attention to the interplay between macroeconomic variables and optimal wage contracts, they do not investigate the inefficiency resulting from these dual roles of the wage contract, which is the main purpose of the present paper.

Masters (2011) analyzes how to provide a correct incentive to invest general human capital in a competitive search model. He argues that if one can commit and advertise the level of human capital and wages, efficient allocation can prevail. Because human capital is general and workers must invest before they meet jobs, there is no trade-off between ex ante allocation and ex post incentive. If a firm can commit a particular wage contract contingent on the level of human capital, they can provide a correct incentive. We show that the results significantly change if human capital is match specific.

More recently, Tsuyuhara (2013) and Lamadon (2014) incorporate a moral hazard problem when a firm must motivate workers to invest in match-specific human capital in order to maintain their relationship in a competitive search framework. They also allow for on-the-job search and derive an increasing wage–tenure profile. However, they do not analyze any welfare implications of their model. Abstracting from several real-world elements, we identify a novel source of inefficiency pertaining to the wage contract that motivates workers to invest in match-specific human capital.

The remainder of the paper is organized as follows. Section 2 introduces the model. Section 3 analyzes the firm's contract-posting problem given the market value of unemployment. Section 4 endogenizes the utility of unemployed workers and solves the market equilibrium. Section 5 analyzes the welfare and shows that when a match-specific skill is critical, the equilibrium wage contract cannot even attain the second best. We also discuss how the misdirected effects distort welfare. Section 6 compares the competitive search model from a search model with wage bargaining and shows that there is some bargaining power under which the search model with wage bargaining can improve welfare. Section 7 shows that if we can introduce up-front fees, the equilibrium contract can attain the first best. The final section concludes and the appendix provides all necessary proofs.

2 Model

In this section, we construct an infinite-horizon discrete-time competitive search model with moral hazard. Firms post their wage contracts and unemployed workers choose a particular submarket characterized by the posted wage contract . Once they find their partner, workers invest in match-specific human capital before production takes place. If they succeed, they are able to produce y_H while at the firm; if they fail, they can only produce y_L while at the firm, where $y_H > y_L \geq 0$.

We assume that if workers fail to invest in match-specific human capital, they are never able to succeed. This seemingly extreme assumption allows us to avoid unnecessary complication of the model and helps clarify our main point. What is important is that workers' effort choices have persistent impacts. When a match-specific investment has a long-run impact, it changes the rent from maintaining their relationship. Hence, it may be optimal for workers to walk away from the firm when they fail to invest in match-specific human capital. Our assumption is designed to capture this mechanism in the simplest possible way.

We describe the behavior of workers and firms at the production stage, an investment stage, and a search stage in order.

Production Stage: After making the investment, both a worker and the firm know how much they can produce (y_H or y_L) from this relationship. Having this information, they decide whether they will continue their relationship. The value from the production relationship at tenure t with the match-specific human capital for workers and firms, W^p_{Ht} and J^p_{Ht}, and without the match-specific human capital for workers and firms, W^p_{Lt} and J^p_{Lt}, is defined as follows:

$$\begin{aligned} W^p_{it} &= w_{it} + \beta \left[\lambda U + (1-\lambda) W^p_{it+1} \right], \\ J^p_{it} &= y_i - w_{it} + \beta \left[\lambda V + (1-\lambda) J^p_{it+1} \right], \end{aligned}$$

where $i = H$ or L, $\lambda \in (0,1)$ is an exogenous separation probability, $\beta \in (0,1)$ is a discount factor, y_i and w_{ti} are the productivity and wage payment for the type i worker at tenure t, and U and V denote the sum of the discounted utilities of unemployed workers and the sum of discounted profit flows from a vacant job, respectively.

Let us define the rent to workers, $R_i \equiv W_{i0}^p - U$, and the surplus from the match, $S_i \equiv J_{i0}^p - V + R_i$. Rearranging the equation, it is easy to see that we can express R_i and S_i as follows:

$$R_i = \sum_{t=0}^{\infty} [\beta (1-\lambda)]^t [w_{it} - (1-\beta) U], \ i = H, L, \tag{1}$$

$$S_i = \frac{y_i - (1-\beta)(U+V)}{1-\beta(1-\lambda)}, \ i = H, L. \tag{2}$$

Because investment takes place only in the initial period, the contract $\mathbf{R} = (R_H, R_L)$ contains all relevant information for the advertisement of the job and the incentive scheme for investment. Hence, we focus below on how firms post $\mathbf{R}$ to attract and motivate workers.

Investment Stage: During the investment stage, the workers invest in match-specific human capital. We assume that they can obtain this capital with probability $e \in [0,1]$ and the monetary cost of the investment is $c(e)$. We make the following standard assumptions concerning the cost function.

Assumption 1: $c'(e) > 0$, $c''(e) > 0$, $c(0) = 0$, $c'(0) = 0$ and $c'(1) = \infty$.

Define $\tilde{R}(R_L) = I(S_L \geq R_L) \max\{R_L, 0\}$ and $\tilde{J}(R_L) = I(R_L \geq 0) \max\{S_L - R_L, 0\}$, where $I(x \geq y) = 1$ if $x \geq y$ and $I(x \geq y) = 0$ if $x < y$. When workers fail to obtain match-specific human capital, there are two possibilities in relation to the firm: to continue the relationship or to separate. Provided that the firm prefers to maintain their relationship, $I(S_L \geq R_L) = 1$, the worker can choose whether to continue the relationship or separate and obtain rent of $\max\{R_L, 0\}$. The value, $\tilde{R}(R_L)$ summarizes the worker's expected rent when he or she fails to obtain the skill. Similarly, provided that the worker prefers to maintain the relationship, $I(R_L \geq 0) = 1$, the firm can choose whether they can continue to maintain their relationship or separate, which provides the firm with rent of $\max\{S_L - R_L, 0\}$. This firm's expected rent is summarized by $\tilde{J}(R_L)$.

Using $\tilde{R}(R_L)$ and $\tilde{J}(R_L)$, the value of being employed workers and occupied jobs before making a match-specific investment, $W_0(\mathbf{R})$ and $J_0(\mathbf{R})$, is defined as follows:

$$\begin{aligned} W_0(\mathbf{R}) &\equiv e(\mathbf{R}) R_H + (1 - e(\mathbf{R})) \tilde{R}(R_L) - c(e(\mathbf{R})) + U, \\ J_0(\mathbf{R}) &\equiv e(\mathbf{R})(S_H - R_H) + (1 - e(\mathbf{R})) \tilde{J}(R_L) + V, \\ e(\mathbf{R}) &= \arg\max_{e \in [0,1]} \left\{ e R_H + (1-e) \tilde{R}(R_L) - c(e) \right\}. \end{aligned}$$

When a worker meets a job, the worker exerts effort and maximizes expected rent net of the cost of effort. With probability $e(\mathbf{R})$, the worker succeeds in obtaining the match-specific human capital and the worker and the firm obtain rent R_H and $S_H - R_H$, respectively. With probability $1 - e(\mathbf{R})$, the worker fails and obtains a rent $\tilde{R}(R_L)$ and the firm obtains a rent $\tilde{J}(R_L)$.

By rearranging the definition of $\tilde{R}(R_L)$ and $\tilde{J}(R_L)$, we rewrite the rent to the worker and the firm when the worker fails to obtain the skill as $\tilde{R}(R_L) = I(S_L \geq R_L \geq 0) R_L$ and $\tilde{J}(R_L) = I(S_L \geq R_L \geq 0)(S_L - R_L)$, where $I(S_L \geq R_L \geq 0) = 1$ if $R_L \in [0, S_L]$, and $I(S_L \geq R_L \geq 0) = 0$ otherwise. This shows that as long as the rent to the failed workers, R_L, is between S_L and 0, the worker obtains rent, R_L, and the firm obtains the remaining surplus, $S_L - R_L$.

Let us define the rent to workers and the surplus from this relationship before undertaking match-specific investment by $R_0(\mathbf{R}) \equiv W_0(\mathbf{R}) - U$ and $S_0(\mathbf{R}) \equiv J_0(\mathbf{R}) - V + R_0(\mathbf{R})$, respectively. Rearranging these equations, we rewrite the investment stage as

$$\begin{aligned} R_0(\mathbf{R}) &= e(\mathbf{R}) \left[R_H - \tilde{R}(R_L) \right] + \tilde{R}(R_L) - c(e(\mathbf{R})), & (3) \\ S_0(\mathbf{R}) &= e(\mathbf{R}) \left[S_H - \tilde{S}(R_L) \right] + \tilde{S}(R_L) - c(e(\mathbf{R})), & (4) \\ c'(e(\mathbf{R})) &= R_H - \tilde{R}(R_L), & (5) \end{aligned}$$

where $\tilde{S}(R_L) = I(S_L \geq R_L \geq 0) S_L$ and $\tilde{R}(R_L) = I(S_L \geq R_L \geq 0) R_L$. Equations (3) and (4) show that the rent to workers, $R_0(\mathbf{R})$, (the surplus from the relationship before making match-specific investment, $S_0(\mathbf{R})$) is the expected rent from the investment, $e(\mathbf{R})\left[R_H - \tilde{R}(R_L)\right] + \tilde{R}(R_L)$, (the expected surplus from the investment, $e(\mathbf{R})\left[S_H - \tilde{S}(R_L)\right] + \tilde{S}(R_L)$) minus the investment cost, $c(e(\mathbf{R}))$. Equation (5) shows that when the worker chooses an optimal effort, the marginal rent from exerting effort, $R_H - \tilde{R}(R_L)$, must be equal to the marginal cost of exerting effort, $c'(e(\mathbf{R}))$. Because the marginal rents depend only on the differences between the rent when the worker succeeds and that when the worker fails, the firm can influence workers' effort by creating appropriate differences in rent for success and failure.

Search Stage: There are many submarkets in an economy and each submarket is characterized by an offered contract, $\mathbf{R}$. Assume that the amount of labor force is normalized to 1. As usual, we assume that there exists a constant returns to scale matching function, $M(v(\mathbf{R}), u(\mathbf{R}))$ for each submarket. Using this matching function, the probability for an available job to meet an unemployed worker is defined by $q(\theta(\mathbf{R})) = M\left(1, \frac{1}{\theta(\mathbf{R})}\right) = \frac{M(v(\mathbf{R}),u(\mathbf{R}))}{v(\mathbf{R})}$, where $\theta(\mathbf{R}) = \frac{v(\mathbf{R})}{u(\mathbf{R})}$. We make the following standard assumption on this function $q(\cdot)$.

Assumption 2: $q(\theta) \in [0,1]$, $q(\theta)\theta \in [0,1]$, $\frac{q'(\theta)\theta}{q(\theta)} \in (-1,0)$, $q''(\theta) < 0$, $q(0) = 1$, $q(\infty) = 0$ and $\lim_{\theta\to 0} \frac{q'(\theta)\theta}{q(\theta)} = 0$.

Using the q function, the probability for a worker to meet a job in the submarket $\mathbf{R}$ can be expressed as $p(\theta(\mathbf{R})) = \frac{M(v(\mathbf{R}),u(\mathbf{R}))}{u(\mathbf{R})} = q(\theta(\mathbf{R}))\theta(\mathbf{R})$. Our assumption on $q(\cdot)$ function ensures the following property of p function: $p(\theta) \in [0,1]$, $p'(\theta) > 0$, $p''(\theta) < 0$, $p(0) = 0$, $p(\infty) = 1$, $p'(0) = 1$, and $p'(\infty) = 0$. We first describe the behavior of unemployed workers and later we describe the behavior of the firm.

Unemployed workers must choose a submarket where they search for a job from the set of offered contracts, $\boldsymbol{\varrho}$. Suppose that the unemployed worker chooses a submarket with $\mathbf{R} \in \boldsymbol{\varrho}$. During the search stage, unemployed workers obtain zero-flow utility and after spending for one period, the unemployed workers find a job with probability $p(\theta(\mathbf{R}))$ and obtain rent $R_0(\mathbf{R})$. Define U^* as the maximum of the following asset equation:

$$\begin{aligned} U^* &= \max_{\mathbf{R} \in \boldsymbol{\varrho}} U(\mathbf{R}), & (6) \\ where\ U(\mathbf{R}) &= \beta\left[p(\theta(\mathbf{R})) R_0(\mathbf{R}) + U\right], & (7) \end{aligned}$$

and $\theta(\mathbf{R})$ must satisfy the following conditions:

$$\frac{U^* - U(\mathbf{R})}{\theta(\mathbf{R})} = 0,\ U^* \geq U(\mathbf{R}),\ \forall \mathbf{R}. \tag{8}$$

This condition ensures that workers do not apply for a submarket with rent $\mathbf{R}$ (even off the equilibrium path) unless it provides them with utility at least equal to U^*.

Firms must post a contract $\mathbf{R}$ before they start searching for workers. The firm must pay a search cost $k > 0$ during this search period and find a worker with probability $q(\theta(\mathbf{R}))$ after one period. When the firm posts the contract, the firm must take into account not only how much the contract attracts unemployed workers, but also the incentive compatibility condition of workers to invest. The firm's contract-posting problem is formulated as follows:

$$\varrho = \arg\max_{\mathbf{R} \in \mathcal{R}^2} \left\{\beta q(\theta(\mathbf{R}))\left[S_0(\mathbf{R}) - R_0(\mathbf{R})\right] - k\right\},\ s.t.\ equation\ (5). \tag{9}$$

Because firms can freely enter the market until their expected profits become 0, the following zero-profit condition must also be satisfied in the equilibrium:

$$V \equiv \beta q(\theta(\mathbf{R}))\left[S_0(\mathbf{R}) - R_0(\mathbf{R})\right] - k = 0, \forall \mathbf{R} \in \boldsymbol{\varrho}. \tag{10}$$

Finally, under a stationary environment, the market value of unemployment, U, must be equal to the highest value of unemployment, U^*:

$$U = U^*. \tag{11}$$

We can now formally define the competitive search equilibrium with an incentive contract.

Definition 1 *A competitive search equilibrium with an incentive contract can be written as an allocation* $\{\theta(\mathbf{R}), e(\mathbf{R}), \varrho, U\}$ *that satisfies the following property.*

1. *Unemployed workers optimally apply for a job: (6), (7) and (8).*

2. *Firms maximize their profits subject to the incentive compatibility (IC) constraint and satisfy the zero-profit condition: (5), (9), and (10).*

3. *The market value of unemployment must be the highest value of unemployment under a stationary equilibrium: (11).*

As Acemoglu and Shimer (1999a) and Guerrieri (2008) show, this problem can be expressed as a rather simple constrained maximization problem and an equilibrium condition.

Proposition 2 $\{\theta(\mathbf{R}), e(\mathbf{R}), \varrho, U\}$ *with* $\theta^* = \theta(\mathbf{R}^*), e^* = e(\mathbf{R}^*)$ *and* $\mathbf{R}^* \in \varrho$ *is a competitive search equilibrium, if and only if,*

1. *for given* U, $(\theta^*, e^*, \mathbf{R}^*)$ *solves*

$$\begin{aligned} U^*(U) &= \max_{e\in[0,1],\mathbf{R}\in\mathcal{R}^2,\theta\in\mathcal{R}_+} \frac{\beta p(\theta) R_0(\mathbf{R})}{1-\beta} \\ s.t.\ c'(e) &= R_H - \tilde{R}(R_L) \\ k &\geq \beta q(\theta)(S_0(\mathbf{R}) - R_0(\mathbf{R})),\ \text{equality if } \theta > 0, \end{aligned}$$

   where $S_0(\mathbf{R}) \equiv e\left(S_H - \tilde{S}(R_L)\right) + \tilde{S}(R_L) - c(e)$, $R_0(\mathbf{R}) \equiv e\left[R_H - \tilde{R}(R_L)\right] + \tilde{R}(R_L) - c(e)$, $\tilde{S}(R_L) = I(S_L \geq R_L \geq 0) S_L$, $\tilde{R}(R_L) = I(S_L \geq R_L \geq 0) R_L$, $p(\theta) = q(\theta)\theta$, *and* $S_i = \frac{y_i - (1-\beta)U}{1-\beta(1-\lambda)}$, *for* $i = H, L$.

2. *the market value of unemployment,* U, *must satisfy the equilibrium condition,* $U = U^*(U)$.

Although this constrained maximization problem is similar to the standard moral hazard problem, there are two main differences. First, unlike the standard moral hazard problem, a principal can choose a labor market condition, θ, and, therefore, the probability of match. Second, because the equilibrium condition endogenously determines this market value of unemployment, there are general equilibrium effects that influence the outcomes of the model. Because of these differences, we show that the model can induce inefficiencies that do not appear in the standard moral hazard problem.

3 The Firm's Contract-posting Problem

In this section, we analyze a firm's contract-posting problem given U. We endogenize U in the following section. Note that by substituting the IC constraint into $R_0(\mathbf{R})$, we can eliminate R_H. Hence, we simplify the firm's contract-posting problem as follows:

$$U^*(U) = \max_{e\in[0,1],\theta\in\mathcal{R}_+,R_L\in\mathcal{R},} \frac{\beta p(\theta)\left[ec'(e)+\tilde{R}(R_L)-c(e)\right]}{1-\beta}$$

subject to the following zero-profit–IC constraint:

$$k \geq \beta q(\theta)\left[e\left[S_H-\tilde{S}(R_L)-c'(e)\right]+\left(\tilde{S}(R_L)-\tilde{R}(R_L)\right)\right], with\ equality\ if\ \theta>0. \tag{12}$$

To guarantee that $\theta > 0$ is feasible, we make the following assumption throughout the paper.

Assumption 3: $\beta\left[e\left[S_H-\tilde{S}(R_L)-c'(e)\right]+\left(\tilde{S}(R_L)-\tilde{R}(R_L)\right)\right] > k$ for some $e\in(0,1)$ and $R_L\in\mathcal{R}$.

The following lemma provides a condition whereby assumption 3 can be feasible.

Lemma 3 *$\frac{y_H}{1-\beta}>U$ if and only if there exists $\hat{k}$ that, for all $k\in\left(0,\hat{k}\right)$, we can find $e>0$ and $R_L\in\mathcal{R}$ that satisfy $k<\beta\left\{e\left[S_H-\tilde{S}(R_L)-c'(e)\right]+\left(\tilde{S}(R_L)-\tilde{R}(R_L)\right)\right\}$.*

Lemma 3 shows that assumption 3 is feasible if and only if $\frac{y_H}{1-\beta}>U$. We assume $\frac{y_H}{1-\beta}>U$ in this section, but later show that in fact an equilibrium value of U also satisfies this assumption.

Let e^*, θ^*, and R_L^* denote the solutions to this contract-posting problem. The following lemma is useful for analyzing our model.

Lemma 4 *For any given* $U < \frac{y_H}{1-\beta}$, (θ^*, R_L^*, e^*) *has the following properties.*

1. *There exist* $\bar{\theta} < \infty$ *such that* $\theta^* \in \left(0, \bar{\theta}\right)$.
2. *If* $S_L > 0$, $R_L^* \in [0, S_L]$. *If* $S_L \leq 0$, *the solution is independent of* R_L^*.
3. $e^* \in (0,1)$ *and* $\frac{d(S_0 - R_0)}{de}|_{e=e^*} = S_H - \hat{S}_L - c'(e^*) - e^* c''(e^*) < 0$.

This lemma 4 shows that θ^* is interior and therefore, the zero-profit–IC constraint is binding. Hence, without loss of generality, we examine the binded zero-profit–IC constraint below. It also shows that as far as there is a surplus to share, it is optimal for a firm to offer the rent that makes it possible to maintain their relationship. Using this lemma, $\tilde{S}(R_L^*) = I(S_L \geq R_L^* \geq 0) S_L = I(S_L \geq 0) S_L$ and $\tilde{R}(R_L^*) = I(S_L \geq R_L^* \geq 0) R_L^* = I(S_L \geq 0) R_L^* \in [0, I(S_L \geq 0) S_L]$. Define $\hat{S}_L = I(S_L \geq 0) S_L$ and $\hat{R}_L = I(S_L \geq 0) R_L \in \left[0, \hat{S}_L\right]$. Because choosing $\hat{R}_L$ is equivalent to choosing R_L, we will consider $\hat{R}_L$ as a choice variable below.

Finally, this lemma 4 shows that e^* is interior and when a firm optimally chooses e, a slight increase in e must lower the firm's ex post profits. Although an increase in e raises the firm's expected surplus, it also increases the expected rent to workers. Hence, the overall impact of an increase in e on expected profits is generally ambiguous. The lemma 4 shows that when a firm optimally encourages a worker's effort, the second effect must dominate the first effect in the equilibrium. To understand the intuition, suppose that an increase in e from an optimal effort e^* raises a firm's ex post profit. This invites more firms to enter and provide more job opportunities and increases the probability of unemployed workers finding a job. Hence, raising e unambiguously increases the value of unemployment, which contradicts the assumption that e^* is an optimal effort.

Using lemma 4, an original contract-posting problem can be rewritten as the following further simplified problem:

$$U^*(U) = \max_{e\in[0,1],\hat{R}_L\in[0,\hat{S}_L],\theta\in[0,\bar{\theta}]} \frac{\beta p(\theta) S_0 - \theta k}{1-\beta}, \tag{13}$$

$$\textit{subject to } k = \beta q(\theta)\left[e\left(S_H - \hat{S}_L - c'(e)\right) + \left(\hat{S}_L - \hat{R}_L\right)\right], \tag{14}$$

where $p(\theta) = q(\theta)\theta$, $S_0 = e\left(S_H - \hat{S}_L\right) + \hat{S}_L - c(e)$, $\hat{S}_L = I(S_L \geq 0) S_L$, and $S_i = \frac{y_i - (1-\beta)U}{1-\beta(1-\lambda)}$ for $i = H$ or L.

Because k is a search cost, $\theta k = \frac{vk}{u}$ is the total search cost per unemployed worker. Therefore, a new problem shows that the firm's contract-posting problem is equivalent to maximizing an ex ante net surplus per unemployed worker subject to the zero-profit–IC constraint (equation (14)).

Note that the objective function is continuous and the choice set is closed and bounded in this revised problem. Hence, the following theorem is immediate.

Theorem 5 *For any given $U < \frac{y_H}{1-\beta}$, there exists a solution to the firm's contract-posting problem.*

To understand this property of the firm's contract-posting problem, we first analyze optimal effort and labor market tightness given U. For this purpose, we consider the following unconstrained ex ante net surplus-maximization problem:

$$\left(e^{best}, \theta^{best}\right) = \arg \max_{e\in[0,1],\theta\in[0,\bar{\theta}]} \frac{\beta p(\theta) S_0 - \theta k}{1-\beta},$$

where $S_0 = e\left[S_H - \hat{S}_L\right] + \hat{S}_L - c(e)$. The first-order conditions are

$$S_H - \hat{S}_L = c'\left(e^{best}\right), \tag{15}$$

$$\beta p'\left(\theta^{best}\right) S_0^{best} = k, \tag{16}$$

where $S_0^{best} = e^{best}\left[S_H - \hat{S}_L\right] + \hat{S}_L - c\left(e^{best}\right)$[3].

Equation (15) shows that an optimal effort equates the marginal cost of effort to the marginal surplus from exerting effort where the marginal surplus is the difference

[3] We can check that there exists a unique $\left(e^{best}, \theta^{best}\right)$ that satisfies the first-order condition and the second-order condition is locally satisfied around $\left(e^{best}, \theta^{best}\right)$.

in surplus between when a worker obtains and does not obtain match-specific human capital. Note that this equation does not depend on the labor market tightness. Hence, irrespective of the market condition, a planner can choose a unique e^{best} for any given U.

Equation (16) shows that optimal labor market tightness equates the marginal benefits of labor market tightness to search cost. Because equation (15) uniquely determines e^{best}, there is a unique S_0^{best} and, therefore, equation (16) can uniquely determine θ^{best}. Define R_0^{best} by R_0 that satisfies the following zero-profit condition, $k = \beta q\left(\theta^{best}\right)\left(S_0^{best} - R_0^{best}\right)$. Rearranging equation (16), we derive a well-known Hosios (1990) condition:

$$\frac{R_0^{best}}{S_0^{best}} = -\frac{q'\left(\theta^{best}\right)\theta^{best}}{q\left(\theta^{best}\right)}.$$

The Hosios (1990) condition shows that social contributions from an increase in θ, $1-\left(-\frac{q'\left(\theta^{best}\right)\theta^{best}}{q\left(\theta^{best}\right)}\right) = \frac{p'\left(\theta^{best}\right)\theta^{best}}{p\left(\theta^{best}\right)}$, which is a percentage increase in workers' probability to find a job by a percentage increase in θ, are equal to the private contributions from an increase in θ, $1-\frac{R_0^{best}}{S_0^{best}}$, which is the fraction of the firm's rent in the surplus that the firm obtains by posting its vacancy.

Define $\hat{R}_L^{best} = S_L - \frac{k}{q\left(\theta^{best}\right)\beta}$. Because equation (16) can uniquely determine θ^{best} for any $U < \frac{y_H}{1-\beta}$, we can also uniquely identify $\hat{R}_L^{best}$ for any $U < \frac{y_H}{1-\beta}$. The following lemma is useful for the characterization of the competitive search equilibrium.

Lemma 6 *For any given $U < \frac{y_H}{1-\beta}$, there exists a unique $S_L^c > 0$ that satisfies*

$$S_L^c = \frac{k}{q\left(\theta^{best}\right)\beta} \in (0, S_H), \ k = \beta p'\left(\theta^{best}\right)\max_e \left\{e\left[S_H - S_L^c\right] + S_L^c - c(e)\right\}.$$

If $S_L \geq S_L^c$, $\hat{R}_L^{best} \geq 0$, and if $S_L < S_L^c$, $\hat{R}_L^{best} < 0$.

Note that $\hat{R}_L \in \left[0, \hat{S}_L\right]$. This lemma shows that there exists a cutoff point of S_L below which $\hat{R}_L^{best}$ is infeasible. Using this lemma, we prove that the following proposition characterizes the solutions to the firm's contract-posting problem.

Proposition 7 *For any given $U < \frac{y_H}{1-\beta}$,*

1. *if* $S_L \geq S_L^c$, *then* $e^* = e^{best}$, $\theta^* = \theta^{best}$, *and* $\hat{R}_L^* = \hat{R}_L^{best}$.

2. *if* $S_L < S_L^c$, *then* $e^* < e^{best}$, $\theta^* < \theta^{best}$, *and* $\hat{R}_L^* = 0$. *Moreover, if* $2c''(e) + ec'''(e) \geq 0$, *the solution is unique.*

To understand the intuition behind this proposition, let us examine whether $\left(e^{best}, \theta^{best}\right)$ satisfies the zero-profit–IC constraint (equation (14)). Note that

$$k = \beta q\left(\theta^{best}\right)\left[e^{best}\left(S_H - \hat{S}_L - c'\left(e^{best}\right)\right) + \hat{S}_L - \hat{R}_L\right] = \beta q\left(\theta^{best}\right)\left(\hat{S}_L - \hat{R}_L\right).$$

Hence, for $\left(e^{best}, \theta^{best}\right)$ to satisfy the zero-profit–IC condition, $\hat{R}_L$ must be equal to $\hat{R}_L^{best}$. Because $\hat{R}_L \in \left[0, \hat{S}_L\right]$, this is possible if and only if $S_L \geq S_L^c$. Note that when $\hat{R}_L^{best}$ is feasible, the wage contracts can handle both the incentive role and the advertisement role at the same time.

$$\begin{aligned} c'\left(e^{best}\right) &= R_H - \hat{R}_L^{best} : \text{ (IC)} \\ q\left(\theta^{best}\right) &= \frac{k}{\beta\left(\hat{S}_L - \hat{R}_L^{best}\right)} : \text{ (0 Profit-IC)} \end{aligned}$$

This shows that $R_H - \hat{R}_L^{best}$ can be chosen to induce e^{best}; $\hat{R}_L^{best}$ can be chosen to target θ^{best}. Hence, it is possible to attain an optimal effort and optimal labor market tightness at the same time.

On the other hand, if $S_L < S_L^c$, $\hat{R}_L^{best} < 0$. Hence, it is impossible to induce an optimal effort. The largest possible punishment for failed workers is to set $\hat{R}_L^* = 0$. Hence, the IC constraint and the zero-profit–IC constraint become

$$\begin{aligned} c'\left(e^*\right) &= R_H, \text{ (IC)} \\ q\left(\theta^*\right) &= \frac{k}{\beta\left[e^*\left(S_H - S_L - c'\left(e^*\right)\right) + \hat{S}_L\right]}. \text{ (0 Profit - IC)} \end{aligned}$$

As this shows, R_H must deal with both e^* and θ^*. That is, the firm must choose R_H to balance the consideration of incentive and advertisement. As a result, both the equilibrium effort and the labor market tightness are less than what is optimal.

Because $S_L^c > 0$, the following corollary is immediate.

Corollary 8 *For any given* $U < \frac{y_H}{1-\beta}$*, if* $S_L < 0$*, then* $e^* < e^{best}$*,* $\theta^* < \theta^{best}$*, and separation occurs when a worker's investment fails. Moreover, if* $2c''(e) + ec'''(e) \geq 0$*, the solution is unique.*

When $S_L < 0$, there is no reason for the firm to maintain a relationship when a worker fails to obtain match-specific human capital. Hence, they decide to separate. Both the equilibrium effort and the labor market tightness are less than what is optimal and the uniqueness is guaranteed by the same condition as before. Hence, we make the further additional assumption to ensure uniqueness below.

Assumption 4: $2c''(e) + ec'''(e) \geq 0$.

4 General Equilibrium

So far, we have analyzed the model for any given $U < \frac{y_H}{1-\beta}$. However, the equilibrium condition must endogenously determine U. In this section, we analyze the equilibrium condition and summarize the characterizations of the competitive search equilibrium. First, the following theorem proves the existence and uniqueness of the equilibrium.

Theorem 9 *There exists a unique* $U \in \left(0, \frac{y_H}{1-\beta}\right)$ *that satisfies* $U = U^*(U)$*.*

Because U is an endogenous variable in the general equilibrium, S_L is also an endogenous variable. Hence, we must restate proposition 7 and corollary 8. The following theorem summarizes the characterizations of the model.

Theorem 10 *There exist unique* $y_L^c \in (y_L^{cc}, y_H)$ *and* $y_L^{cc} \in (0, y_L^c)$*, where*

1. *for all* $y_L \in [y_L^c, y_H)$*,* $e^* = e^{best}$*,* $\theta^* = \theta^{best}$*, and* $R_L^* = R_L^{best} \geq 0$*.*
2. *for all* $y_L \in [y_L^{cc}, y_L^c)$*,* $e^* < e^{best}$*,* $\theta^* < \theta^{best}$*, and* $R_L^* = 0$*.*
3. *for all* $y_L \in [0, y_L^{cc})$*,* $e^* < e^{best}$*,* $\theta^* < \theta^{best}$*, and separation occurs when a worker's investment fails.*

We measure the criticality of match-specific human capital for production by the level of y_L. If y_L is large, the job can be productive without the match-specific human capital. Hence, the match-specific human capital is not critical. In contrast, a small y_L means that the productivity of jobs without the match-specific human capital is fairly small. That is, the match-specific human capital is critical.

The theorem shows that if match-specific human capital is not critical for production, the contract can induce optimal effort and optimal labor market tightness where optimal means that it maximizes the ex ante net surplus of unemployed workers.

However, if it is critical, the effort and labor market tightness are lower than optimal. There are two possibilities. If $y_L \in [y_L^{cc}, y_L^c)$, they can maintain their relationship. But because a firm offers $R_L^* = 0$, a worker does not obtain any rent from this relationship. If $y_L < y_L^{cc}$, they decide to separate. Hence, the value of employed workers who fail to obtain match-specific human capital is the same as the value of unemployed workers in both cases. In other words, the equilibrium contract coincides with a version of an efficiency wage contract.

5 Welfare Analysis

In this section, we analyze the welfare property of the competitive search equilibrium. In the previous section, we define $\left(e^{best}, \theta^{best}\right)$ as an effort level and labor market tightness that maximize unconstrained ex ante net surplus per unemployed worker. This is potentially a good welfare criterion for a partial equilibrium analysis. However, once we wish to consider social welfare for a whole economy, this criterion may not be appropriate because the economy consists not only of unemployed workers, but also employed workers. The social planner must then take into account resource constraints in society and maximize a reasonable social welfare function.

First, we describe the resource constraints in an economy. Let $n_{H,t}^B$ and $n_{H,t}^E$ denote the fraction of workers who succeed in obtaining the match-specific human capital at the beginning and the end of date t, respectively. Similarly, let $n_{L,t}^B$ and $n_{L,t}^E$ denote the fraction of workers who fail to obtain the match-specific human capital at the beginning and the end of date t, and u_t^B and u_t^E denote the fraction of unemployed workers at the beginning and the end of date t, respectively. Finally, let n_{0t}^B denote the fraction of matched workers who have not yet invested in match-specific human capital at the beginning of date t. The following dynamics of employment status summarize the resource constraints in an economy:

$$n_{H,t+1}^B = (1-\lambda) n_{H,t}^E, \; n_{L,t+1}^B = (1-\lambda) n_{L,t}^E, \; n_{0,t+1}^B = p(\theta_t) u_t^E, \tag{17}$$

$$u_{t+1}^B = (1 - p(\theta)) u_t^E + \lambda \left(n_{H,t}^E + n_{L,t}^E\right), \tag{18}$$

$$n_{H,t}^E = e_t n_{0,t}^B + n_{H,t}^B, \; n_{L,t}^E = x_t (1-e_t) n_{0,t}^B + n_{L,t}^B, \tag{19}$$

$$u_t^E = (1-x_t)(1-e_t) n_{0,t}^B + u_t^B, \tag{20}$$

where $x_t \in [0,1]$ is the probability that maintains their relationship when a worker fails to obtain match-specific human capital. When the fraction of employed workers with and without the match-specific human capital is $n_{H,t}^E$ and $n_{L,t}^E$ at the end of date t, $(1-\lambda)$ portion of them can maintain their status and λ portion of them become unemployed at the beginning of date $t+1$. When the fraction of unemployed workers is u_t^E at the end of date t, $1-p(\theta)$ portion of them remain unemployed and $p(\theta)$ portion of them find a potential new job at the beginning of date $t+1$. These dynamics are described in equations (17) and (18). When the fraction of potential employees at the beginning of date t is n_{0t}^B, e_t portion of them succeed in obtaining the match-specific human capital and $(1-e_t)$ portion of them fail to obtain it. Hence, $e_t n_{0,t}^B + n_{H,t}^B$ becomes the fraction of employed workers with the match-specific human capital at the end of date t, $x_t(1-e_t) n_{0,t}^B + n_{L,t}^B$ becomes the fraction of employed without it, and $(1-x_t)(1-e_t) n_{0,t}^B + u_t^B$ becomes the fraction of unemployed workers at the end of date t. Equations (19) and (20) formally describe these processes.

We assume that a social planner maximizes the sum of the discounted stream of net output, where the net output consists of aggregate output, $y_H n_{H,t}^E + y_L n_{L,t}^E$ minus aggregate search cost, $k\theta_t u_t^E = kv_t$ and the aggregate cost of investment $c(e_t) n_{0,t}^B$. Denote $\mathbf{N}_t^B = \left(n_{H,t}^B, n_{L,t}^B, n_{0,t}^B, u_t^B\right)$ and $\mathbf{N}_t^E = \left(n_{H,t}^E, n_{L,t}^E, u_t^E\right)$. The planner's first-best problem is expressed by the following Bellman equation:

$$Y^F\left(\mathbf{N}_t^B\right) = \max_{x_t\in[0,1],e_t\in[0,1],\theta_t\in\left[0.\bar{\theta}\right]} \left\{ \begin{array}{c} y_H n_{H,t}^E + y_L n_{L,t}^E - k\theta_t u_t^E \\ -c\left(e_t\right) n_{0,t}^B + \beta Y^F\left(\mathbf{N}_{t+1}^B\right) \end{array} \right\}, \tag{21}$$

subject to equations (17), (18), (19), and (20). Define a function $S_i\left(U\right) \equiv \frac{y_i-(1-\beta)U}{1-\beta(1-\lambda)}$ where $i = H$ or L and $\hat{S}_L\left(U\right) = I\left(S_L\left(U\right) \geq 0\right) S_L\left(U\right)$. The following lemma is useful to understand the property of the social planner problem.

Lemma 11 *The planner's first-best problem can be simplified by*

$$Y^F\left(\mathbf{N}_t^B\right) = S_H\left(U^F\right) n_{H,t}^B + S_L\left(U^F\right) n_{L,t}^B + S_0\left(U^F\right) n_{0,t}^B + U^F,$$

where $S_0\left(\cdot\right)$ and U^F are solutions to the following equations:

$$\begin{aligned} U^F &= \max_{\theta\in\left[0,\bar{\theta}\right]} \frac{p\left(\theta\right)\beta S_0\left(U^F\right) - \theta k}{1-\beta}, \\ S_0\left(U^F\right) &= \max_{e\in[0,1]} \left\{ e\left(S_H\left(U^F\right) - \hat{S}_L\left(U^F\right)\right) + \hat{S}_L\left(U^F\right) - c\left(e\right)\right\}. \end{aligned}$$

We would like to compare the first-best problem and the solution to the competitive search equilibrium. The following lemma makes the comparison easier.

Lemma 12 *The solutions to the competitive search equilibrium are equivalent to the solutions to the following problem:*

$$\begin{aligned} U^* &= \max_{\theta\in\left[0,\bar{\theta}\right]} \frac{p\left(\theta\right)\beta S_0\left(\theta, U^*\right) - \theta k}{1-\beta}, \\ & S_0\left(\theta, U^*\right) = \max_{e\in[0,1],\hat{R}_L\in\left[0,\hat{S}_L\right]} \left\{ eS_H\left(U^*\right) + \left(1-e\right)\hat{S}_L\left(U^*\right) - c\left(e\right)\right\}, \\ & s.t.\ k = \beta q\left(\theta\right)\left[e\left(S_H\left(U^*\right) - \hat{S}_L\left(U^*\right) - c'\left(e\right)\right) + \left(\hat{S}_L - \hat{R}_L\right)\right]. \end{aligned} \tag{22}$$

Set $e_t = e^*\left(\theta^*, U^*\right)$, $\theta_t = \theta^*\left(U^*\right)$, and $x_t = I\left(S_L\left(U^*\right) \geq 0\right)$, where $e^*\left(\cdot,\cdot\right)$, $\theta^*\left(\cdot\right)$ and U^* are solutions to the problem (22). Evaluate the welfare under a competitive search equilibrium, $Y^*\left(\mathbf{N}_t^B\right)$, by the value function in equation (21) without maximization, together with resource constraints (17), (18), (19), and (20). It can be shown by a proof similar to that of lemma 11 that $Y^*\left(\mathbf{N}_t^B\right) = S_H\left(U^*\right) n_{H,t}^B +$

$S_L(U^*)\,n_{L,t}^B + S_0(\theta^*(U^*), U^*)\,n_{0,t}^B + U^*$, where $S_0(\cdot,\cdot)$, $\theta^*(\cdot)$, and U^* are the solutions to the problem (22). Hence, it is apparent from lemma 11 and lemma 12 that the only difference between the first-best problem and the competitive search equilibrium is the existence of the zero-profit–IC constraint in the competitive search equilibrium. From the results from theorem 10, the following proposition is immediate.

Proposition 13 *There exists y_L^c such that*

1. *for all $y_L \geq y_L^c$, e and θ in the first-best and the competitive search equilibrium are the same.*

2. *for all $y_L < y_L^c$, e and θ are lower in the competitive search equilibrium than those in the first-best equilibrium.*

When $y_L \geq y_L^c$, because a firm can arbitrarily adjust $\hat{R}_L$, the zero-profit–IC constraint does not cause any meaningful restriction on the choice of effort and labor market tightness. However, when $y_L < y_L^c$, $\hat{R}_L = 0$. Hence, the zero-profit–IC constraint becomes $k = \beta q(\theta)\left[e\left(S_H(U^*) - \hat{S}_L(U^*) - c'(e)\right) + \hat{S}_L\right]$. Note that the total differentiation of this equation shows $\frac{\partial \theta}{\partial e} < 0$. That is, the zero-profit–IC constraint causes a trade-off between the effort and the workers' probability of finding a job; namely, a firm must pay a higher wage to induce greater effort. However, because the promise of a high wage reduces the firms' profits, only a small number of firms can enter the market, which reduces labor market tightness and, therefore, the unemployed workers' probability of finding a job. That is, when $y_L < y_L^c$, it is impossible to increase both e and $p(\theta)$ at the same time.

It is natural to ask whether a planner can attain a better allocation than that under the competitive search equilibrium when the zero-profit–IC constraint is satisfied. To examine this question, we assume that a planner can tax labor market tightness at the rate τ and finance unemployment benefits z. That is, we substitute $U(\mathbf{R}) = z - \theta(\mathbf{R})\,\tau + \beta\left[p(\theta(\mathbf{R}))\,R_0(\mathbf{R}) + U\right]$ for equation (7) and impose a budget constraint $z = \tau \int \theta(\mathbf{R})\,dQ(\mathbf{R})$ as an additional constraint, where Q is an equilibrium distribution of $\mathbf{R}$. This labor market tightness tax is designed to clarify the main source of inefficiency in the competitive search model. The only role of this tax is to lead unemployed workers to choose that labor market with less labor market tightness than that in the competitive search equilibrium. We show that a slight increase in labor market tightness tax from zero can improve welfare.

Using the same argument as before, the welfare under the competitive search equilibrium with labor market tightness tax τ is summarized by

$$Y\left(\tau:\mathbf{N}_t^B\right)=S_H\left(U^\tau\right)n_{H,t}^B+S_L\left(U^\tau\right)n_{L,t}^B+S_0\left(\theta^\tau;U^\tau\right)n_{0,t}^B+U^\tau,$$

where $S_0\left(\cdot;\cdot\right)$, θ^τ, and U^τ are the solutions to

$$\begin{aligned} U^\tau &= \hat{U}\left(\theta^\tau,U^\tau\right)\equiv\frac{\beta p\left(\theta^\tau\right)S_0\left(\theta^\tau;U^\tau\right)-k\theta^\tau}{1-\beta}, \\ \theta^\tau &= \theta\left(\tau,U^\tau\right)\equiv\arg\max_{\theta\in\left[0,\bar{\theta}\right]}\frac{z-\theta\tau+p\left(\theta\right)\beta S_0\left(\theta;U^\tau\right)-\theta k}{1-\beta}, \end{aligned}$$

$$\begin{aligned} S_0\left(\theta;U^\tau\right) &= \max\nolimits_{e\in[0,1],\hat{R}_L\in\left[0,\hat{S}_L\right]}\left\{e\left(S_H\left(U^\tau\right)-\hat{S}_L\left(U^\tau\right)\right)+\hat{S}_L\left(U^\tau\right)-c\left(e\right)\right\}, \\ s.t.\ k &= \beta q\left(\theta\right)\left[e\left(S_H\left(U^\tau\right)-\hat{S}_L\left(U^\tau\right)-c'\left(e\right)\right)+\hat{S}_L\left(U^\tau\right)-\hat{R}_L\right]. \end{aligned} \tag{23}$$

Note that θ^τ maximizes ex ante surplus plus $\frac{z-\theta\tau}{1-\beta}$, but τ does not directly influence U^τ because the budget constraint imposes $z=\theta\tau$. Therefore, τ can influence welfare only through the changes in θ^τ.

Proposition 14 *There exists* $U^\tau\in\left(0,\frac{y_H}{1-\beta}\right)$ *that satisfies* $B\left(U^\tau\right)\equiv U^\tau-\hat{U}\left(\theta\left(\tau,U^\tau\right),U^\tau\right)=0$*. Moreover, there exists* $\hat{\tau}\in(0,\infty]$ *such that for any* $\hat{\tau}>\tau\geq 0$*,* U^τ *is unique. For such* τ*,*

$$\begin{aligned} \frac{dY\left(\tau:\mathbf{N}_t^B\right)}{d\tau} &= \left[(C.S.)+(M.D.)+(N.E.)\right]\frac{\partial\theta\left(\tau,U^\tau\right)}{\partial\tau} \qquad (24) \\ (C.S.) &= \frac{\beta\lambda+(1-\beta)u_t^E}{1-\beta(1-\lambda)}\frac{\frac{\partial\hat{U}\left(\theta^\tau,U^\tau\right)}{\partial\theta^\tau}}{B'\left(U^\tau\right)}\geq 0, \\ (M.D.) &= n_{0,t}^B\left[S_H\left(U^\tau\right)-\hat{S}_L\left(U^\tau\right)-c'\left(e^\tau\right)\right]\frac{\partial e\left(\theta^\tau,U^\tau\right)}{\partial\theta^\tau}\frac{\frac{d\theta\left(\tau,U^\tau\right)}{d\tau}}{\frac{\partial\theta\left(\tau,U^\tau\right)}{\partial\tau}}\leq 0, \\ (N.E.) &= n_{0,t}^B\left[S_H\left(U^\tau\right)-S_L\left(U^\tau\right)-c'\left(e^\tau\right)\right]\frac{\partial e\left(\theta^\tau,U^\tau\right)}{\partial U^\tau}\frac{\frac{\partial\hat{U}\left(\theta^\tau,U^\tau\right)}{\partial\theta^\tau}}{B'\left(U^\tau\right)}\leq 0, \end{aligned}$$

where $e^{\tau} = e(\theta^{\tau}, U^{\tau})$ *is an optimal solution of* e *to the problem (23),* $\frac{\partial \hat{U}(\theta^{\tau},U)}{\partial \theta^{\tau}} = \frac{\tau}{1-\beta} \geq 0$, $\frac{\frac{d\theta(\tau,U^{\tau})}{d\tau}}{\frac{\partial \theta(\tau,U^{\tau})}{\partial \tau}} = 1 + \frac{\frac{\partial \theta(\tau,U^{\tau})}{\partial U^{\tau}} \frac{\partial \hat{U}(\theta^{\tau},U^{\tau})}{\partial \theta^{\tau}}}{B'(U^{\tau})} > 0$, $\frac{\partial \theta(\tau,U)}{\partial \tau} < 0$, $B'(U) > 0$ *and* $\frac{\partial e(\theta,U)}{\partial U} \leq 0$, $\frac{\partial e(\theta,U)}{\partial \theta} \leq 0$ *with a strict inequality if* $e < e^{best}$.

Proposition 14 shows that a rise in τ lowers labor market tightness, θ^{τ} and that the reduction in θ^{τ} has three different effects on welfare.

The first term, $(C.S.)$, captures the standard property of the competitive search model. Because $\frac{\beta\lambda+(1-\beta)u_0^B}{[1-\beta(1-\lambda)]B'(U^{\tau})} > 0$, the sign of $(C.S.)$ is the same as the sign of $\frac{\partial \hat{U}(\theta^{\tau},U^{\tau})}{\partial \theta^{\tau}}$. Because $\frac{\partial \hat{U}(\theta^{\tau},U^{\tau})}{\partial \theta^{\tau}} \geq 0$, $(C.S.)$ is nonnegative and, therefore, τ can reduce social welfare through $(C.S.)$. If $e^{\tau} = e^{best}$, then $S_H(U^{\tau}) - S_L(U^{\tau}) = c'(e^{\tau})$ and $(M.D.) = (N.E.) = 0$. In this case, $\frac{dY(\tau:\mathbf{N}_t^B)}{d\tau} = 0$ if and only if $\frac{\partial \hat{U}(\theta^{\tau},U^{\tau})}{\partial \theta^{\tau}} = \frac{\tau}{1-\beta} = 0$. Hence, if $e^{\tau} = e^{best}$, the competitive search equilibrium ($\tau = 0$) maximizes social welfare.

If $e^{\tau} < e^{best}$, then $S_H(U^{\tau}) - \hat{S}_L(U^{\tau}) > c'(e^{\tau})$ and a planner must consider two different effects. The second term, $(M.D.)$, represents the impact of a change in θ on the ex post surplus, $S_0(\theta^{\tau}, U^{\tau})$, through $\frac{\partial e(\theta^{\tau},U^{\tau})}{\partial \theta^{\tau}}$, because when $e^{\tau} < e^{best}$, the zero-profit–IC constraint introduces the trade-off between effort and labor market tightness, $\frac{\partial e(\theta^{\tau},U^{\tau})}{\partial \theta^{\tau}} < 0$. Knowing that $\frac{\frac{d\theta(\tau,U^{\tau})}{d\tau}}{\frac{\partial \theta(\tau,U^{\tau})}{\partial \tau}} > 0$, if $e^{\tau} < e^{best}$, then $(M.D.)$ is negative. This means that an increase in τ increases welfare through $(M.D.)$.

The intuition can be understood by the following logic. When $e^{\tau} < e^{best}$, the firm can encourage greater effort by providing more rent to successful workers. However, because transferring rent to workers reduces profits, the number of posted job offers will be lower. This lowers the job-finding probability of unemployed workers and thus lowers the value of unemployed workers. Hence, this type of submarket cannot survive in a competitive economy. Thus, when a wage must play an advertisement and an incentive role at the same time, the competition to attract workers forces firms to offer a wage to improve the ex ante utility of workers at the expense of their ex post utility. We refer to this as misdirected effect. By leading unemployed workers to choose a labor market with less labor market tightness than that under the competitive search equilibrium, a planner can mitigate the misdirected effect. This is why an increase in τ increases welfare through $(M.D.)$.

The third term, $(N.E)$, represents the effect of changes in τ on ex post surplus, $S_0(\theta^\tau, U^\tau)$ through $\frac{\partial e(\theta^\tau, U^\tau)}{\partial U^\tau}$. Note that, when $e^\tau < e^{best}$, $\hat{R}_L = 0$ and therefore, the incentive compatibility condition becomes $R_H = c'(e^*)$. Because a rise in U^τ lowers the surplus from the relationship, a firm must reduce the rent to workers R_H in order to avoid negative profits, which in turn lowers the effort of workers, $\frac{\partial e(\theta^\tau, U^\tau)}{\partial U^\tau} < 0$. This means that increases in the utility of unemployed workers due to the competition to attract workers make it costly for other firms to provide workers with appropriate incentives. We call this a negative externality effect. Because $\frac{\frac{\partial \hat{U}(\theta^\tau, U^\tau)}{\partial \theta^\tau}}{B'(U^\tau)}$ is positive, if $e^\tau < e^{best}$, $(N.E.)$ is negative. This means that an increase in τ increases welfare through $(N.E.)$.

Although the overall effects are ambiguous, because $\frac{\partial \hat{U}(\theta^\tau, U^\tau)}{\partial \theta^\tau} = \frac{\tau}{1-\beta}$, when $\tau = 0$, this term is 0. Hence, if $e^\tau < e^{best}$,

$$\frac{dY\left(\tau : \mathbf{N}_t^B\right)}{d\tau}|_{\tau=0} = n_{0,t}^B \left[S_H\left(U^\tau\right) - \hat{S}_L\left(U^\tau\right) - c'\left(e^\tau\right)\right] \frac{\partial e\left(\theta^\tau, U^\tau\right)}{\partial \theta} \frac{\partial \theta\left(\tau, U^\tau\right)}{\partial \tau} > 0.$$

The result can be summarized by the following proposition.

Proposition 15 *Suppose that $y_L < y_L^c$. A slight increase in τ under a competitive search equilibrium improves welfare because it mitigates the misdirected effects.*

The proposition shows that when match-specific human capital is critical, the competition to attract workers obliges firms to offer wage contracts that cause too many firms to enter the market. A tax on labor market tightness mitigates this distortion and improves welfare.

6 Competitive Search Model vs. Search Model with Wage Bargaining

In this section, we compare the welfare under a competitive search model and a search model with wage bargaining. As far as a competitive search model attains the constrained optimal, the welfare under a search model with wage bargaining that faces the same constraint cannot be greater than that under the competitive search model. However, because the competitive search model can fail to attain the constrained optimal in our model, there is an opportunity for a search model with wage bargaining to improve social welfare. We examine this possibility.

Following the standard assumption on a search model with wage bargaining, we assume that R_H and R_L can be determined by a generalized Nash bargaining: $R_H = \alpha S_H(U)$ and $R_L = \alpha S_L(U)$, where α is the bargaining power of workers. Note that this wage determination policy leads to $\tilde{R}(R_L) = I(S_L(U) \geq R_L \geq 0) R_L = \alpha \hat{S}_L(U)$ and $\tilde{S}(R_L) = I(S_L(U) \geq R_L \geq 0) S_L(U) = \hat{S}_L(U)$. Hence, the following IC condition uniquely determines the effort level, $e^B(\alpha, U)$:

$$c'\left(e^B(\alpha, U)\right) = \alpha\left(S_H(U) - \hat{S}_L(U)\right). \tag{25}$$

Similarly, we can identify labor market tightness $\theta^B(\alpha, U)$ using the zero-profit–IC condition:

$$q\left(\theta^B(\alpha, U)\right) = \frac{k}{\beta\left[\begin{array}{c} e^B(\alpha, U)\left(S_H(U) - \hat{S}_L(U) - c'\left(e^B(\alpha, U)\right)\right) \\ +\hat{S}_L(U) - \alpha\hat{S}_L(U) \end{array}\right]}. \tag{26}$$

We can now express social welfare under a search model with wage bargaining by

$$Y^B\left(\mathbf{N}_t^B\right) = S_H\left(U^B\right) n_{H,t}^B + S_L\left(U^B\right) n_{L,t}^B + S_0^B\left(\alpha, U^B\right) n_{0,t}^B + U^B,$$

where $S_0^B(\cdot,\cdot)$ and U^B are solutions to the following equations:

$$\begin{aligned} U^B &= \frac{p\left(\theta^B\left(\alpha, U^B\right)\right)\beta S_0^B\left(\alpha, U^B\right) - k\theta^R\left(\alpha, U^B\right)}{1-\beta}, \\ S_0^B\left(\alpha, U^B\right) &= e^B\left(\alpha, U^B\right) S_H\left(U^B\right) + \left(1 - e^B\left(\alpha, U^B\right)\right)\hat{S}_L\left(U^B\right) - c\left(e^B\left(\alpha, U^B\right)\right), \end{aligned}$$

where $e^B(\alpha, U)$ satisfies equation (25) and $\theta^B(\alpha, U)$ satisfies (26). The following proposition shows that it is possible for a search model with wage bargaining to attain higher welfare than the competitive search model.

Proposition 16 *For all $y_L < y_L^{cc}$, there exists α, which attains higher welfare than that under the competitive search equilibrium.*

The intuition of the proof can be explained as follows. When $y_L < y_L^{cc}$, the separation occurs when a worker fails to obtain a match-specific skill. Hence, $R_L = \alpha \hat{S}_L\left(U^B\right) = 0$ and $R_H = \alpha S_H\left(U^B\right)$. In this case, we can always find α^* so that $\alpha^* S_H\left(U^B\right) = R_H^*$ where R_H^* is the solution to the competitive search equilibrium. Because we know that a planner can improve welfare by slightly lowering θ and increasing e from the competitive search equilibrium, we can find $\alpha = \alpha^* + \varepsilon$ where $\varepsilon > 0$, under which a search model with wage bargaining improves welfare.

7 Up-Front Fees

In this section, we show that when we allow up-front fees, the competitive search equilibrium can always attain the first-best equilibrium. This exercise clarifies the role of limited liability in deriving our results.

Suppose that a worker must pay up-front fees, $w_f \geq 0$, to the matched firm before investing in the match-specific human capital. Then the value of being employed workers and occupied jobs before making match-specific investment can be modified as follows:

$$
\begin{aligned}
W_0(\mathbf{R}) &\equiv e(\mathbf{R}) R_H + (1 - e(\mathbf{R})) \tilde{R}(R_L) - c(e(\mathbf{R})) - w_f + U, \\
J_0(\mathbf{R}) &\equiv e(\mathbf{R}) (S_H - R_H) + (1 - e(\mathbf{R})) \tilde{J}(R_L) + w_f + V,
\end{aligned}
$$

where $e(\mathbf{R}) = \arg\max_e \left\{ e R_H + (1-e) \tilde{R}(R_L) - c(e) \right\}$. Following the same steps as above, we can rewrite our contract-posting problem as follows:

$$
\begin{aligned}
U &= \max_{e \in [0,1], \hat{R}_L \in \left[0, \hat{S}_L\right], \theta \in \left[0, \bar{\theta}\right], w_f \geq 0} \frac{\beta p(\theta) S_0 - \theta k}{1 - \beta}, \\
S_0 &= e\left[S_H(U) - \hat{S}_L(U)\right] + \hat{S}_L(U) - c(e), \\
k &= q(\theta) \beta \left\{ e\left[S_H(U) - \hat{S}_L(U) - c'(e)\right] + \hat{S}_L(U) - \hat{R}_L + w_f \right\}.
\end{aligned}
$$

Now, a firm has an additional choice variable, $w_f \geq 0$. Hence, it is accurate that even if $\hat{R}_L = 0$, a firm can choose $w_f > 0$ and induce an optimal effort and an optimal labor market tightness by appropriately choosing R_H and w_f.

$$\begin{aligned} c'\left(e^{best}\right) &= R_H : \text{ (IC)} \\ q\left(\theta^{best}\right) &= \frac{k}{\beta\left(\hat{S}_L\left(U\right) + w_f\right)} : \text{ (zero-profit–IC)} \end{aligned}$$

Following this observation, we can now safely claim the following proposition.

Proposition 17 *If a firm can charge up-front fees, the competitive search model can always induce an optimal effort and an optimal labor market tightness.*

If up-front fees are acceptable, workers are willing to pay the fees if the firm promises a sufficiently high wage when they succeed in investing in match-specific human capital. This is feasible because if this rearrangement induces optimal effort, it can generate greater surplus. In other words, up-front fees can be considered as a transfer mechanism from ex ante surplus to ex post surplus to motivate workers to exert an effort. If this transfer mechanism exists, the wage contract that maximizes unemployed workers also maximizes social welfare. This result indicates that the lack of the transfer mechanism because of limited liability is necessary to derive our results.

8 Conclusion

This paper analyzes an equilibrium wage contract when a firm must motivate workers to invest in match-specific human capital in a competitive search model. We examine when and how the dual roles of a wage contract, advertisement and motivation, interact with each other. Our model identifies a novel source of inefficiency, which we call *misdirected effect*: when a wage must play two different roles, the competition to attract workers forces a wage to be chosen to increase the ex ante utility of workers at the expense of ex post utility, which induces too many job openings and makes it unprofitable for firms to pay a higher wage to motivate workers. Because of this effect, a competitive search model cannot attain the constrained optimum. We also show that there are possibilities that a search model with ex post bargaining can improve welfare.

The paper also suggests that entry fees can be a possible solution to the problem. For this exercise, we implicitly assume that there is no liquidity constraint and they can pay fees in advance. This may not be feasible in reality[4]. One possible solution may be implicit bonding, such as deferred or seniority wages, as in Lazear (1979). This could serve as a criticism of an efficiency wage because the implicit bonding eliminates unemployed workers. But because of search friction, the unemployed worker still exists under a seniority wage scheme in our model. Hence, our model can be consistent with the coexistence of seniority wages and unemployed workers in the same market.

Nevertheless, note that seniority wages serve as a solution only if the accumulation of match-specific human capital takes some time. If workers can quickly accumulate match-specific human capital, lower payments to workers when they are young may not be enough to cover all required up-front fees to attain the first-best allocation. In this case, we literally need up-front fees to solve the inefficiency addressed in this paper. We allowed only an initial investment in skill to make this point clear. The dynamic accumulation of match-specific human capital may be interesting for a quantitative assessment of deferred wages, which is left for our future research.

9 Appendix

Proof of Proposition 2: We first show the following lemma. Then, we show that the competitive search equilibrium solves the constrained maximization problem. Later we prove the opposite direction.

Lemma 18 *Suppose that* $\{\theta(\mathbf{R}), e(\mathbf{R}), \varrho, U\}$ *is the competitive search equilibrium, then* $U(\mathbf{R}) = \frac{\beta p(\theta(\mathbf{R}))R_0(\mathbf{R})}{1-\beta}$.

Proof. *Define* $\mathbf{R}^* = \arg\max_{\mathbf{R}\in\varrho} U(\mathbf{R})$. *Then it is easy to see that* $\mathbf{R}^* = \arg\max_{\mathbf{R}\in\varrho} \frac{\beta p(\theta(\mathbf{R}))R_0(\mathbf{R})}{1-\beta}$.

[4] Guerrieri (2008) explicitly considers a feasibility constraint for up-front fees in her model

Because $U^* = U$ *on the equilibrium,*

$$U^* = \beta\left[p\left(\theta\left(\mathbf{R}^*\right)\right)R_0\left(\mathbf{R}^*\right) + U\right] = \frac{\beta p\left(\theta\left(\mathbf{R}^*\right)\right)R_0\left(\mathbf{R}^*\right)}{1-\beta} = \max_{\mathbf{R}\in\boldsymbol{\varrho}} \frac{\beta p\left(\theta\left(\mathbf{R}\right)\right)R_0\left(\mathbf{R}\right)}{1-\beta}.$$

For any $\mathbf{R}$*, it is easy to check that* $\frac{U^*-U(\mathbf{R})}{\theta(\mathbf{R})} = 0$ *if and only if* $\frac{U^* - \frac{\beta p(\theta(\mathbf{R}))R_0(\mathbf{R})}{1-\beta}}{\theta(\mathbf{R})} = 0.$ ■

Necessity: Let $\{\theta\left(\mathbf{R}\right), e\left(\mathbf{R}\right), \varrho, U\}$ be a competitive search equilibrium with $\mathbf{R}^* \in \varrho$, $\theta^* = \theta\left(\mathbf{R}^*\right)$, and $e^* = e\left(\mathbf{R}^*\right)$. We must prove that $\{\mathbf{R}^*, \theta^*, e^*\}$ solves the constrained optimization problem. Because it satisfies the zero-profit condition and the IC condition,

$$k = \beta q\left(\theta^*\right)\left(S_0^* - R_0^*\right), \ R_H^* - \tilde{R}\left(R_L^*\right) = c'\left(e\left(\mathbf{R}^*\right)\right) = c'\left(e^*\right),$$

where $S_0^* \equiv e^*\left(S_H - \tilde{S}\left(R_L^*\right)\right) + \tilde{S}\left(R_L^*\right) - c\left(e^*\right)$ and $R_0^* \equiv e^*\left[R_H^* - \tilde{R}\left(R_L^*\right)\right] + \tilde{R}\left(R_L^*\right) - c\left(e^*\right)$. Suppose that another triple $\{\mathbf{R}, \theta, e\}$ satisfies the IC condition and achieves a higher value of the objective. That is,

$$U < \frac{\beta p\left(\theta\right)R_0}{1-\beta} = \frac{\beta p\left(\theta\right)\left\{e\left[R_H - \tilde{R}\left(R_L\right)\right] + \tilde{R}\left(R_L\right) - c\left(e\right)\right\}}{1-\beta}.$$

We shall prove that it must violate the zero-profit condition. Take this $\mathbf{R} = (R_H, R_L)$ and consider $\{\theta\left(\mathbf{R}\right), e\left(\mathbf{R}\right), \varrho, U\}$. Because both $e\left(\mathbf{R}\right)$ and e satisfy the IC condition, $e\left(\mathbf{R}\right) = e$. Hence, optimal application implies that

$$U \geq \frac{\beta p\left(\theta\left(\mathbf{R}\right)\right)R_0\left(\mathbf{R}\right)}{1-\beta} = \frac{\beta p\left(\theta\left(\mathbf{R}\right)\right)\left\{e\left[R_H - \tilde{R}\left(R_L\right)\right] + \tilde{R}\left(R_L\right) - c\left(e\right)\right\}}{1-\beta}.$$

This implies that $\theta > \theta\left(\mathbf{R}\right) \geq 0$. Therefore,

$$\beta q\left(\theta\right)\left[S_0 - R_0\right] < \beta q\left(\theta\left(\mathbf{R}\right)\right)\left[S_0 - R_0\right] = \beta q\left(\theta\left(\mathbf{R}\right)\right)\left[S_0\left(\mathbf{R}\right) - R_0\left(\mathbf{R}\right)\right] = k.$$

Because $\theta > 0$, this violates the zero-profit condition.

Sufficiency: We now prove by construction that for any solution $\{\mathbf{R}^*, \theta^*, e^*\}$ to the constrained maximization problem, there is a solution to the competitive search equilibrium $\{\theta\left(\mathbf{R}\right), e\left(\mathbf{R}\right), \varrho, U\}$ with $\varrho = \{\mathbf{R}^*\}$, $\theta^* = \theta\left(\mathbf{R}^*\right)$, and $e^* = e\left(\mathbf{R}^*\right)$. Let $e\left(\mathbf{R}\right)$ satisfy $c'\left(e\left(\mathbf{R}\right)\right) = R_H - \tilde{R}\left(R_L\right)$ for all $\mathbf{R}$, set $U = \frac{\beta p(\theta^*)\left\{e^*\left[R_H^* - \tilde{R}\left(R_L^*\right)\right] + \tilde{R}\left(R_L^*\right) - c(e^*)\right\}}{1-\beta}$

and choose $\theta(\mathbf{R})$ to satisfy $U = \frac{\beta p(\theta(\mathbf{R}))\left\{e(\mathbf{R})\left[R_H - \tilde{R}(R_L)\right] + \tilde{R}(R_L) - c(e(\mathbf{R}))\right\}}{1-\beta}$ or $\theta(\mathbf{R}) = \infty$ if there is no solution to the equation. It is immediate that $\{\theta(\mathbf{R}), e(\mathbf{R}), \varrho, U\}$ satisfies an optimal application. We now show that it also satisfies the zero-profit condition. Suppose to the contrary that some triples $\{\mathbf{R}', \theta(\mathbf{R}'), e(\mathbf{R}')\}$ violate the zero-profit condition. Because it implies that $\beta q(\theta(\mathbf{R}'))[S_0(\mathbf{R}') - R_0(\mathbf{R}')] > k$, $\theta(\mathbf{R}') < \infty$. Therefore, there exists $\tilde{\theta} > \theta(\mathbf{R}')$ such that $\beta q\left(\tilde{\theta}\right)[S_0(\mathbf{R}') - R_0(\mathbf{R}')] = k$ and

$$\begin{aligned} U &= \frac{\beta p(\theta(\mathbf{R}'))\left\{e(\mathbf{R}')\left[R'_H - \tilde{R}(R'_L)\right] + \tilde{R}(R'_L) - c(e(\mathbf{R}'))\right\}}{1-\beta} \\ &< \frac{\beta p\left(\tilde{\theta}\right)\left\{e(\mathbf{R}')\left[R'_H - \tilde{R}(R'_L)\right] + \tilde{R}(R'_L) - c(e(\mathbf{R}'))\right\}}{1-\beta}. \end{aligned}$$

Hence, $\left(\mathbf{R}', \tilde{\theta}, e(\mathbf{R}')\right)$ attain a higher objective function than $\{\mathbf{R}^*, \theta^*, e^*\}$. Contradiction. **Q.E.D.**

Proof of Lemma 3: Suppose that $\frac{y_H}{1-\beta} > U$. Choose $R_L < 0$. Then $e\left[S_H - \tilde{S}(R_L) - c'(e)\right] + \left(\tilde{S}(R_L) - \tilde{R}(R_L)\right) = e(S_H - c'(e))$. Because $\frac{y_H}{1-\beta} > U$, $S_H > 0$. Therefore, because $c'(e) = R_H \in (0, S_H)$, there exists e such that $e(S_H - c'(e)) > 0$. Therefore, we can find $\hat{k}$ that for all $k \in \left(0, \hat{k}\right)$, $\beta e(S_H - c'(e)) > k$. On the contrary, suppose that $\frac{y_H}{1-\beta} \leq U$. Because $y_H > y_L$, $S_H \leq 0$ and $\tilde{S}(R_L) = \tilde{R}(R_L) = 0$. Hence, $\beta\left[e\left[S_H - \tilde{S}(R_L) - c'(e)\right] + \left(\tilde{S}(R_L) - \tilde{R}(R_L)\right)\right] = \beta e[S_H - c'(e)] \leq 0$. Therefore, it is impossible to find $e > 0$ and $R_L \in \mathcal{R}$ that satisfy $k < \beta e[S_H - c'(e)]$ for all $k > 0$. **Q.E.D.**

Proof of Lemma 4:

1. Suppose there is no $\bar{\theta} < \infty$ such that $\theta^* \leq \bar{\theta}$. Then $\theta^* = \infty$. It violates the zero-profit condition. Suppose $\theta^* = 0$. Then $p(0) = 0$ and $U = 0$. Take $e > 0$ and R_L such that $\beta\left[e\left[S_H - \tilde{S}(R_L) - c'(e)\right] + \left(\tilde{S}(R_L) - \tilde{R}(R_L)\right)\right] > k$, and choose θ so that $q(\theta) = \frac{k}{\beta\left[e\left[S_H - \tilde{S}(R_L) - c'(e)\right] + \left(\tilde{S}(R_L) - \tilde{R}(R_L)\right)\right]} < 1$. Hence, $\theta > 0$.

Because $\frac{dR_0}{de} = \frac{d[ec'(e)-c(e)]}{de} = ec''(e) > 0$, $\frac{\beta p(\theta)\left[ec'(e)-c(e)+\tilde{R}(R_L)\right]}{1-\beta} > 0 = U$. This contradicts the assumption that $\theta^* = 0$.

2. If $S_L > 0$ and $R_L^* \notin [0, S_L]$, then $\tilde{S}(R_L^*) = 0$ and $\tilde{R}(R_L^*) = 0$. Hence, (e^*, θ^*) must satisfy $U = \frac{\beta p(\theta^*)[e^* c'(e^*)-c(e^*)]}{1-\beta}$ and $k = q(\theta^*)\beta e^*(S_H - c'(e^*))$. It means that $e^* \in (0,1)$. Choose $R_L' = 0$. Then $\tilde{S}(R_L') = S_L$ and $\tilde{R}(R_L') = 0$. Because $e^* \in (0,1)$, we can find $\theta_0 > \theta^*$ such that $q(\theta_0) = \frac{k}{\beta[e^*(S_H - c'(e^*)) + (1-e^*)S_L]}$. Note that $U_0 = \frac{\beta p(\theta_0)[e^* c'(e^*)-c(e^*)]}{1-\beta} > U$. Contradiction. Suppose that $S_L \leq 0$. Because $I(S_L \geq R_L \geq 0) = 0$ for all R_L, the result is obvious.

3. Let us first show that $e^* \in (0,1)$. Suppose that $e^* = 1$. Because $c'(1) = \infty$, it violates the zero-profit–IC condition. Suppose that $e^* = 0$. Then $U = \frac{\beta p(\theta^*)\hat{R}_L^*}{1-\beta}$ and $q(\theta^*) = \frac{k}{\beta\left[\hat{S}_L - \hat{R}_L^*\right]}$. Because $y_H > y_L$, $S_H > \hat{S}_L$. Hence, there exists ε and $\theta_\varepsilon > \theta^*$ such that $q(\theta_\varepsilon) = \frac{k}{\beta\left[\varepsilon\left(S_H - \hat{S}_L - c'(\varepsilon)\right) + \hat{S}_L - \hat{R}_L^*\right]}$. But $\theta_\varepsilon > \theta^*$ implies that $\frac{\beta p(\theta_\varepsilon)\left[\varepsilon c'(\varepsilon) - c(\varepsilon) + \hat{R}_L^*\right]}{1-\beta} > U$. Contradiction. Suppose that $\frac{d(S_0 - R_0)}{de}|_{e=e^*} > 0$. Then $\frac{dq}{de}|_{e=e^*} = -\frac{k}{\beta(S_0-R_0)^2}\frac{d(S_0-R_0)}{de}|_{e=e^*} < 0$. Therefore, if we define $\tilde{p}(q(\theta)) = p(\theta)$, $\frac{d\tilde{p}(q)}{de}|_{e=e^*} > 0$ and $\frac{dR_0}{de} = ec''(e) > 0$. This implies that a slight increase in e from e^* improves the objective function. Contradiction. **Q.E.D.**

Proof of Lemma 6: Consider G function: $G(S_L) = S_L - \frac{k}{q(\theta^{best})\beta}$ where $k = \beta p'(\theta^{best}) S_0^{best}$ and $S_0^{best} = \max_e \{e[S_H - S_L] + S_L - c(e)\}$. If $S_L = 0$, $G(0) = -\frac{k}{q(\theta^{best})\beta} < 0$. If $S_L = S_H - \varepsilon > 0$, then $G(S_H - \varepsilon) = S_H - \varepsilon - \frac{p'(\theta^{best})\left[e^{best}\varepsilon + S_H - \varepsilon - c(e^{best})\right]}{q(\theta^{best})}$. Hence, it can be shown that $G(S_H - \varepsilon) > -\frac{q'(\theta^{best})\theta^{best}}{q(\theta^{best})}(S_H - \varepsilon) - \frac{p'(\theta^{best})e^{best}\varepsilon}{q(\theta^{best})}$ and we can find ε such that $G(S_H - \varepsilon) > 0$. Therefore, there exists $S_L^c \in (0, S_H)$. Note that

$$G'(S_L) = 1 + \frac{kq'(\theta^{best})}{\beta q(\theta^{best})^2}\frac{d\theta^{best}}{dS_0^{best}}\left(1 - e^{best}\right).$$

Because of $\beta p'\left(\theta^{best}\right)S_0^{best}=k$, $\frac{d\theta^{best}}{dS_0^{best}}=-\frac{p'\left(\theta^{best}\right)}{p''\left(\theta^{best}\right)S_0^{best}}>0$. Hence, we can obtain

$$\begin{aligned}1+\frac{kq'\left(\theta^{best}\right)}{\beta q\left(\theta^{best}\right)^2}\frac{d\theta^{best}}{dS_0^{best}} &= 1-\frac{q'\left(\theta^{best}\right)}{p''\left(\theta^{best}\right)}\left[1+\frac{q'\left(\theta^{best}\right)\theta^{best}}{q\left(\theta^{best}\right)}\right]^2 \\ &> 1-\frac{q'\left(\theta^{best}\right)}{p''\left(\theta^{best}\right)}=\frac{q'\left(\theta^{best}\right)+q''\left(\theta^{best}\right)\theta^{best}}{p''\left(\theta^{best}\right)}>0.\end{aligned} \tag{27}$$

Therefore, $G'\left(S_L\right)>0$ and S_L^c is unique. The desired result follows from the definition of $\hat{R}_L^{best}$. **Q.E.D.**

Proof of Proposition7: Before proving the proposition, we first prove the following lemma:

Lemma 19 *$e^*\leq e^{best}$.*

Proof. *Suppose not. Because we know $e^*\in(0,1)$, there exists $e^*\in\left(e^{best},1\right)$, $\hat{R}_L^*\in\left[0,\hat{S}_L\right]$ and $\theta^*\in\left[0,\bar{\theta}\right]$ such that $k=\beta q\left(\theta^*\right)\left[e^*\left(S_H-\hat{S}_L-c'\left(e^*\right)\right)+\left(\hat{S}_L-\hat{R}_L^*\right)\right]$. Note that because $c''(e)>0$, $c'\left(e^*\right)>c'\left(e^{best}\right)$ and $S_H-\hat{S}_L<c'\left(e^*\right)$. Suppose $\hat{S}_L=\hat{R}_L^*$. Then $q\left(\theta^*\right)=\frac{k}{\beta e^*\left[S_H-\hat{S}_L-c'\left(e^*\right)\right]}<0$. Contradiction. Hence, $\hat{S}_L>\hat{R}_L^*$. Therefore, there exist $\varepsilon>0$ and $\hat{R}_L^\varepsilon\in\left(\hat{R}_L^*,\hat{S}_L\right)$ such that $\beta q\left(\theta^*\right)\left[\begin{array}{c}\left(e^*-\varepsilon\right)\left(S_H-\hat{S}_L-c'\left(e^*-\varepsilon\right)\right)\\+\left(\hat{S}_L-\hat{R}_L^\varepsilon\right)\end{array}\right]=k$. Take this ε. The mean value theorem implies that there exist $\hat{\varepsilon}\in(0,\varepsilon)$*

$$\begin{aligned}&\left(e^*-\varepsilon\right)\left[S_H-\hat{S}_L\right]+\hat{S}_L-c\left(e^*-\varepsilon\right),\\ =\;& e^*\left[S_H-\hat{S}_L\right]+\hat{S}_L-c\left(e^*\right)-\left[S_H-\hat{S}_L-c'\left(e^*\right)\right]\varepsilon+\frac{c''\left(\hat{\varepsilon}\right)}{2}\varepsilon^2,\\ >\;& e^*\left[S_H-\hat{S}_L\right]+\hat{S}_L-c\left(e^*\right).\end{aligned}$$

Contradiction. ■

$\mathbf{S}_L\geq\mathbf{S}_L^c$: Suppose that $S_L\geq S_L^c$. Substituting $\left(e^{best},\theta^{best}\right)$ into the zero-profit–IC constraint,

$$k=q\left(\theta^{best}\right)\beta\left[e^{best}\left[S_H-\hat{S}_L-c'\left(e^{best}\right)\right]+\left(\hat{S}_L-\hat{R}_L^*\right)\right]=q\left(\theta^{best}\right)\beta\left(S_L-\hat{R}_L^*\right).$$

This means that $\hat{R}_L^* = S_L - \frac{k}{q\left(\theta^{best}\right)\beta} = \hat{R}_L^{best}$. Because $S_L \geq S_L^c$, $\hat{R}_L^{best} \geq 0$. That is, if a firm sets $\hat{R}_L^* = \hat{R}_L^{best}$, $\left(e^{best}, \theta^{best}\right)$ is attainable. By the definition of $\left(e^{best}, \theta^{best}\right)$, this must be the solution to the problem.

$\mathbf{S}_L < \mathbf{S}_L^c$: Suppose that $S_L \in (0, S_L^c)$. Because we know e^* and θ^* are interior, the first-order conditions must be characterized by

$$0 = \frac{\beta p'(\theta^*) S_0|_{e=e^*} - k}{1-\beta} - \mu\beta q'(\theta^*)(S_0 - R_0)|_{e=e^*}, \tag{28}$$

$$0 = \frac{\beta p(\theta^*) \frac{dS_0}{de}|_{e=e^*}}{1-\beta} - \mu\beta q(\theta^*) \frac{d(S_0 - R_0)}{de}|_{e=e^*}, \tag{29}$$

$$0 = \mu\beta + \lambda_1 - \lambda_2,\ 0 = \lambda_1 \hat{R}_L^*,\ \lambda_1 \geq 0,\ 0 = \lambda_2\left(\hat{S}_L - \hat{R}_L^*\right),\ \lambda_2 \geq 0,$$

where μ is the Lagrange multiplier for the zero-profit–IC constraint, λ_1 is the multiplier for the nonnegative constraint of $\hat{R}_L$, and λ_2 is the multiplier for the upper bound constraint of $\hat{R}_L$.

First, we show that $e^* < e^{best}$. Suppose not. Because $e^* \leq e^{best}$, $e^* = e^{best}$. Hence, $\frac{dS_0}{de} = 0$. Because $\frac{d(S_0-R_0)}{de}|_{e=e^*} < 0$, $\mu = 0$ from equation (29). This means that $\beta p'(\theta^*) S_0 = k$ from equation (28) and therefore, that $\theta^* = \theta^{best}$. Substituting $e^* = e^{best}$ and $\theta^* = \theta^{best}$ into the (zero-profit–IC) constraint, we obtain $\hat{R}_L^* = \hat{S}_L - \frac{k}{q\left(\theta^{best}\right)\beta}$. But this is not feasible. Contradiction.

Next, we show that $\hat{R}_L^* = 0$. Suppose not. Then $\lambda_1 = 0$ and, therefore, $\mu = \frac{\lambda_2}{\beta} \geq 0$. But because $e^* < e^{best}$, $\frac{dS_0}{de}|_{e=e^*} > 0$ and $\frac{d(S_0-R_0)}{de}|_{e=e^*} < 0$. Hence, $\mu < 0$ from equation (29). Contradiction.

Finally, we show that $\theta^* < \theta^{best}$. Because $e^* < e^{best}$ and $\mu \neq 0$, equations (28) and (29) imply

$$\beta p(\theta^*) \frac{dS_0}{de}|_{e=e^*} = \frac{\beta p'(\theta^*) S_0|_{e=e^*} - k}{q'(\theta^*)} \frac{\beta q(\theta^*)^2}{k} \frac{d(S_0 - R_0)}{de}|_{e=e^*}.$$

Because $\frac{dS_0}{de}|_{e=e^*} > 0$ and $\frac{d(S_0-R_0)}{de}|_{e=e^*} < 0$, the equation implies that $k < \beta p'(\theta^*) S_0|_{e=e^*}$. By definition, $S_0 \leq S_0^{best}$. Therefore,

$$\beta p'(\theta^*) S_0^{best} \geq \beta p'(\theta^*) S_0|_{e=e^*} > k = \beta p'\left(\theta^{best}\right) S_0^{best}.$$

Because $p''(\theta) < 0$, $\theta^* < \theta^{best}$.

Suppose that $S_L \leq 0$. Then $\hat{S}_L = \hat{R}_L^* = 0$ and the first-order condition must be characterized by equations (28) and (29). Applying the same argument as above, $e^* < e^{best}$ and $\theta^* < \theta^{best}$.

To prove the uniqueness of e^* and θ^*, let us define $D(e)$ and $\tilde{\theta}(e)$, which satisfy the following two equations:

$$\begin{aligned} D(e) &= \beta p\left(\tilde{\theta}(e)\right)\frac{dS_0}{de} - \frac{\beta p'\left(\tilde{\theta}(e)\right)S_0 - k}{q'\left(\tilde{\theta}(e)\right)}\frac{\beta q\left(\tilde{\theta}(e)\right)^2}{k}\frac{d(S_0 - R_0)}{de}, \\ q\left(\tilde{\theta}(e)\right) &= \frac{k}{\beta[S_0 - R_0]}. \end{aligned}$$

Note that when the first-order conditions are satisfied, $D(e) = 0$. From the second equation, we show that $\tilde{\theta}'(e) = -\frac{\beta q\left(\tilde{\theta}(e)\right)^2}{kq'\left(\tilde{\theta}(e)\right)}\frac{d(S_0-R_0)}{de} < 0$. Substituting this into the first equation, we find that

$$D(e) = \beta p\left(\tilde{\theta}(e)\right)\frac{dS_0}{de} + \left[\beta p'\left(\tilde{\theta}(e)\right)S_0 - k\right]\tilde{\theta}'(e).$$

Taking the derivative around $D(e) = 0$, we derive

$$\begin{aligned} D'(e)|_{D(e)=0} &= \beta p'(\theta^*)\tilde{\theta}'(e^*)\frac{dS_0}{de}|_{e=e^*} + \beta p(\theta^*)\frac{d^2S_0}{de^2}|_{e=e^*} \\ &\quad + \beta p''(\theta^*)S_0|_{e=e^*}\left[\tilde{\theta}'(e^*)\right]^2 + \left[\beta p'(\theta^*)S_0|_{e=e^*} - k\right]\tilde{\theta}''(e^*). \end{aligned}$$

Note that $\frac{dS_0}{de}|_{e=e^*} > 0$, $\frac{d^2S_0}{de^2}|_{e=e^*} < 0$, and $\beta p'(\theta^*)S_0|_{e=e^*} > k$. Hence, if $\tilde{\theta}''(e^*) < 0$, then $D'(e)|_{D(e)=0} < 0$. By taking the second derivative, we show that

$$\tilde{\theta}''(e) = \frac{\left[2\frac{q'\left(\tilde{\theta}(e)\right)\tilde{\theta}(e)}{q\left(\tilde{\theta}(e)\right)} - \frac{q''\left(\tilde{\theta}(e)\right)\tilde{\theta}(e)}{q'\left(\tilde{\theta}(e)\right)}\right]\frac{\beta^2 q\left(\tilde{\theta}(e)\right)^4}{\tilde{\theta}(e)}\left(\frac{d(S_0-R_0)}{de}\right)^2 - \beta q\left(\tilde{\theta}(e)\right)^2\frac{d^2(S_0-R_0)}{de^2}kq'\left(\tilde{\theta}(e)\right)}{k^2 q'\left(\tilde{\theta}(e)\right)^2}.$$

Because we show that $\frac{d^2(S_0-R_0)}{de^2} = -[2c''(e) + ec'''(e)] \leq 0$, $\tilde{\theta}''(e) < 0$ and, therefore, $D'(e)|_{D(e)=0} < 0$. This means that if there exists e^*, it is unique. Given e^*, $q(\theta^*) = \frac{k}{\beta[S_0-R_0]|_{e=e^*}}$ derives a unique θ^*. Because we know that there is a solution to the original problem and the solution must satisfy $D(e) = 0$, this must be the unique solution to the original problem. **Q.E.D.**

Proof of Theorem 9: Define $F(U) = U - U^*(U)$. Because the objective function of equation (13) is continuous and the choice set is compact, the theorem of maximum implies that $F(U)$ is continuous. Suppose $U = \frac{y_H - \varepsilon}{1-\beta}$, where $\varepsilon > 0$. Then $S_H = \frac{\varepsilon}{1-\beta(1-\lambda)}$. Because $y_H > y_L$, we can choose ε so that $\frac{y_L}{1-\beta} < U = \frac{y_H-\varepsilon}{1-\beta}$. Then $S_0 = e^* \frac{\varepsilon}{1-\beta(1-\lambda)} - c(e^*)$. So we can choose ε so that $U^*\left(\frac{y_H-\varepsilon}{1-\beta}\right)$ are close to 0. That is, we can find $\varepsilon > 0$ so that $F\left(\frac{y_H-\varepsilon}{1-\beta}\right) > 0$. Suppose $U = 0$. Note that the contract-posting problem can be rewritten as

$$
\begin{aligned}
U^*(U) &= \max_{e\in[0,1],\hat{R}_L\in\left[0,\hat{S}_L\right]} \frac{\beta\tilde{p}(q)\left[ec'(e) - c(e) + \hat{R}_L\right]}{1-\beta}, \\
q &= \frac{k}{\beta\left[e\left[S_H - \hat{S}_L - c'(e)\right] + \left(\hat{S}_L - \hat{R}_L\right)\right]},
\end{aligned} \tag{30}
$$

where $\tilde{p}(q(\theta)) = p(\theta)$. Because $e^* > 0$ for all $U < \frac{y_H}{1-\beta}$, it is easy to see that $U^*(0) > 0$. Therefore, $F(0) < 0$. This proves the existence. Taking the derivative of F with respect to U by using equation (30),

$$
F'(U) = 1 + \frac{\beta\tilde{p}'(q)R_0}{1-\beta}\frac{dq}{d(S_0 - R_0)}\frac{(1-\beta)}{1-\beta(1-\lambda)}\left[e^* + (1-e^*)I[S_L \geq 0]\right] > 0,
$$

for all $S_L \neq 0$. This proves the uniqueness of $F(U) = 0$. **Q.E.D.**

Proof of Theorem 10**:** Define $y_L^c(U)$ and U^* such that $S_L^c = \frac{y_L^c(U)-(1-\beta)U}{(1-\beta(1-\lambda))}$ and $U^* = U^*(U^*)$. Because lemma 6 shows that $S_L^c \in (0, S_H)$ and theorem 9 shows that $U^* \in \left(0, \frac{y_H}{1-\beta}\right)$, we can show $0 < (1-\beta)U^* < y_L^c(U^*) < y_H$ for any given y_L. From the definition of $y_L^c(U^*)$, and proposition 7 and corollary 8, if we can take U^* as given, the desired result is immediate by setting $y^c = y_L^c(U^*)$ and $y^{cc} = (1-\beta)U^*$. We have to show that the result does not change even if we take into account the endogeneity of U^*. Let us define $\Phi_1(y_L) = y_L - (1-\beta)U^*$, $\Phi_2(y_L) = y_L - y_L^c(U^*)$, $F(U^* : y_L) = 0$, and $G(S_L^c : U^*) = 0$, where $F(U : y_L) = U - U^*(U)$ and $G(S_L : U) = S_L - \frac{k}{q\left(\theta^{best}\right)\beta}$. Because of $0 < (1-\beta)U^* < y_L^c(U^*) < y_H$ for any $y_L \in (0, y_H)$, it is clear that $\Phi_1(0) < 0$, $\Phi_2((1-\beta)U^*) < 0$, $\Phi_1((1-\beta)U^*) > 0$, and $\Phi_2(y_H - \varepsilon) > 0$ for small $\varepsilon > 0$. Because $\Phi_1(y_L)$, $\Phi_2(y_L)$, $F(U : y_L)$ and $G(S_L : U)$ are continuous, there exists $y^c = y_L^c(U^*) \in \left((1-\beta)U^*, \frac{y_H}{1-\beta}\right)$ and $y^{cc} = (1-\beta)U^* \in (0, y_L^c(U^*))$. To obtain the desired results, we need to prove $\Phi_1'(y_L) > 0$ and $\Phi_2'(y_L) > 0$. Note that

$$\begin{aligned}
\Phi_1'(y_L) &= 1 - \frac{d(1-\beta)U}{dy_L}|_{U=U^*}, \\
\Phi_2'(y_L) &= 1 - \left[1 + (1-\beta(1-\lambda))\frac{dS_L^c}{d(1-\beta)U}|_{S_L=S_L^c}\right]\frac{d(1-\beta)U}{dy_L}|_{U=U^*}.
\end{aligned}$$

We can show that

$$\begin{aligned}
\frac{d(1-\beta)U}{dy_L}|_{U=U^*} &= -(1-\beta)\frac{F_{y_L}(U^*;y_L)}{F_u(U^*;y_L)} \\
&= \frac{\frac{\beta\tilde{p}'(q)R_0}{1-\beta}\frac{dq}{d(S_0-R_0)}\frac{(1-\beta)I(S_L\geq 0)(1-e)}{1-\beta(1-\lambda)}}{1+\frac{\beta\tilde{p}'(q)R_0}{1-\beta}\frac{dq}{d(S_0-R_0)}\frac{(1-\beta)[e+(1-e)I[S_L\geq 0]]}{1-\beta(1-\lambda)}} \in (0,1).
\end{aligned}$$

This proves $\Phi_1'(y_L) > 0$. Because $\beta p'\left(\theta^{best}\right)\max_e\{e[S_H - S_L^c] + S_L^c - c(e)\} = k$, we can also show that

$$\frac{dS_L^c}{d(1-\beta)U}|_{S_L=S_L^c} = -\frac{1}{(1-\beta)}\frac{G_U(S_L:U)|_{S_L=S_L^c}}{G_{S_L}(S_L:U)|_{S_L=S_L^c}} = \frac{\frac{kq'\left(\theta^{best}\right)}{\beta q\left(\theta^{best}\right)^2}\frac{d\theta^{best}}{dS_0^{best}}e^{best}\frac{1}{1-\beta(1-\lambda)}}{1+\frac{kq'\left(\theta^{best}\right)}{\beta q\left(\theta^{best}\right)^2}\frac{d\theta^{best}}{dS_0^{best}}\left(1-e^{best}\right)},$$

and, therefore, that

$$1 + (1-\beta(1-\lambda))\frac{dS_L^c}{d(1-\beta)U}|_{S_L=S_L^c} = \frac{1+\frac{kq'\left(\theta^{best}\right)}{\beta q\left(\theta^{best}\right)^2}\frac{d\theta^{best}}{dS_0^{best}}}{1+\frac{kq'\left(\theta^{best}\right)}{\beta q\left(\theta^{best}\right)^2}\frac{d\theta^{best}}{dS_0^{best}}\left(1-e^{best}\right)}.$$

Note that equation (27) shows that $1+\frac{kq'\left(\theta^{best}\right)}{\beta q\left(\theta^{best}\right)^2}\frac{d\theta^{best}}{dS_0^{best}} > 0$. Hence, $1+(1-\beta(1-\lambda))\frac{dS_L^c}{d(1-\beta)U}|_{S_L=S_L^c} \in (0,1)$. Therefore, $\Phi_2'(y_L) > 0$, and the desired results follow. **Q.E.D.**

Proof of Lemma 11: Suppose that $Y\left(\mathbf{N}_t^B\right) = T_H^{FB}n_{H,t}^B + T_L^{FB}n_{L,t}^B + T_0^{FB}n_{0,t}^B + U^{FB}u_t^B$. Then the Bellman equation can be rewritten as

$$\begin{aligned}
&\max_\theta\left\{y_H n_{H,t}^E + y_L n_{L,t}^E - k\theta u_t^E + \beta\left(T_H^{FB}n_{H,t+1}^B + T_L^{FB}n_{L,t+1}^B + T_0^{FB}n_{0,t+1}^B + U^{FB}u_{t+1}^B\right)\right\} \\
= \; & T_H^{FE}n_{H,t}^E + T_L^{FE}n_{L,t}^E + U^{FE}u_t^E,
\end{aligned}$$

where $T_H^{FE} \equiv y_H + \beta\left[T_H^{FB}(1-\lambda) + \lambda U^{FB}\right]$, $T_L^{FE} \equiv y_L + \beta\left[T_L^{FB}(1-\lambda) + \lambda U^{FB}\right]$, and $U^{FE} \equiv \max_\theta \left\{\beta\left[T_0^{FB} p(\theta) + U^{FB}(1-p(\theta))\right] - k\theta\right\}$, and

$$\max_{e,x}\left\{-c(e_t)\, n_{0,t}^B + \left[T_H^{FE} n_{H,t}^E + T_L^{FE} n_{L,t}^E + U^{FE} u_t^E\right]\right\}$$
$$= \left\{\begin{array}{c} T_H^{FE} n_{H,t}^B + T_L^{FE} n_{L,t}^B + U^{FE} u_t^B \\ + \max_{e,x}\left\{e_t T_H^{FE} + (1-e_t)\left[x_t T_L^{FE} + (1-x_t) U^{FE}\right] - c(e_t)\right\} n_{0,t}^B \end{array}\right\}.$$

Hence, T_H^{FB}, T_L^{FB}, T_0^{FB} and U^{FB} must satisfy

$$\begin{aligned} T_i^{FB} &= y_i + \beta\left[(1-\lambda) T_i^{FB} + \lambda U^{FB}\right],\ i = H, L, \\ T_0^{FB} &= \max_{e,x}\left\{e_t T_H^{FB} + (1-e_t)\left[x_t T_L^{FB} + (1-x_t) U^{FB}\right] - c(e_t)\right\}, \\ U^{FB} &= \max_\theta\left\{\beta\left[T_0^{FB} p(\theta) + U^{FB}(1-p(\theta))\right] - k\theta\right\}. \end{aligned}$$

Because the Bellman equation is a contraction mapping, the solution to this equation must be a unique solution to the original Bellman equation. Define $S_i^F = T_i^{FB} - U^{FB}$ and $U^F = U^{FB}$. Then we can find $S_i^F = \frac{y_i - (1-\beta)U^F}{1-\beta(1-\lambda)}$, where $i = H$ and L, $S_0^F = \max_{e,x}\left\{e_t S_H^F + (1-e_t)\, x_t S_L^F - c(e_t)\right\}$ and $U^F = \max_\theta \frac{p(\theta)\beta S_0^F - k\theta}{1-\beta}$. Note that $x_t = 1$ if and only if $S_L^F \geq 0$. The desired results follow. **Q.E.D.**

Proof of Lemma 12**:** Let $\left\{e^+, \hat{R}_L^+, \theta^+, U^+\right\}$ be a solution to the contract-posting problem under a competitive search equilibrium for any given U and let $\left\{e^{++}(\theta^{++}), \hat{R}_L^{++}(\theta^{++}), \theta^{++}, U^{++}\right\}$ be a solution to the maximization problem in the problem (22) for any given U. Suppose that there exists $\left\{e^+, \hat{R}_L^+, \theta^+, U^+\right\}$ that does not solve the maximization problem of (22). This means that because $\left\{e^+, \hat{R}_L^+, \theta^+\right\}$ must satisfy the zero-profit–IC condition, for any $U < \frac{y_H}{1-\beta}$, $S_0\left(\theta^+, U\right) > e^+\left[S_H(U) - \hat{S}_L(U)\right] + \hat{S}_L(U) - c(e^+)$ or $\max_{\theta\in\left[0,\bar{\theta}\right]} \frac{\beta p(\theta) S_0(\theta,U) - \theta k}{1-\beta} > \frac{\beta p\left(\theta^+\right) S_0\left(\theta^+, U\right) - \theta^+ k}{1-\beta}$. For both cases,

$$\begin{aligned} U^+ &= \frac{\beta p\left(\theta^+\right)\left[e^+\left[S_H(U) - \hat{S}_L(U)\right] + \hat{S}_L(U) - c(e^+)\right] - \theta^+ k}{1-\beta} \\ &< \max_{\theta\in\left[0,\bar{\theta}\right]} \frac{\beta p(\theta) S_0(\theta, U) - \theta k}{1-\beta} = U^{++}. \end{aligned}$$

Contradiction. On the contrary, suppose that there exists $\left\{e^{++}\left(\theta^{++}\right), \hat{R}_L^{**}\left(\theta^{++}\right), \theta^{++}, U^{++}\right\}$ that does not solve the maximization problem in a competitive search equilibrium. This means that because $\left\{e^{++}\left(\theta^{++}\right), \hat{R}_L^{++}\left(\theta^{++}\right), \theta^{++}\right\}$ satisfies the zero-profit–IC condition, there exists $\left\{e^{+}, \hat{R}_L^{+}, \theta^{+}, U^{+}\right\}$ that satisfies $U^{+} > \max_{\theta \in \left[0, \bar{\theta}\right]} \frac{\beta p(\theta) S_0(\theta) - \theta k}{1-\beta}$. This means that

$$\begin{aligned} U^{+} &> \max_{\theta \in \left[0, \bar{\theta}\right]} \frac{\beta p\left(\theta\right) S_0\left(\theta\right) - \theta k}{1-\beta} \geq \frac{\beta p\left(\theta^{+}\right) S_0\left(\theta^{+}\right) - \theta^{+} k}{1-\beta} \\ &\geq \frac{\beta p\left(\theta^{+}\right)\left[e^{+}\left[S_H - \hat{S}_L\right] + \hat{S}_L - c\left(e^{+}\right)\right] - \theta^{+} k}{1-\beta} = U^{*}. \end{aligned}$$

Contradiction. Because e^{+}, $\hat{R}_L^{+}$, and θ^{+} are unique for any $U < \frac{y_H}{1-\beta}$, $\left\{e^{+}, \hat{R}_L^{+}, \theta^{+}, U^{+}\right\} = \left\{e^{++}\left(\theta^{++}\right), \hat{R}_L^{++}\left(\theta^{++}\right), \theta^{++}, U^{++}\right\}$ for any $U < \frac{y_H}{1-\beta}$. Hence, the equivalence must also hold for the value of the unemployed under the competitive search equilibrium. Because we know the competitive search equilibrium has a unique solution, the desired results follow. **Q.E.D.**

Proof of Proposition 14**:** Because the objective function is continuous and the choice set is compact, the theorem of maximum implies that $S_0\left(\theta, U\right)$ is continuous in θ and U. The theorem of maximum also implies that $\theta\left(\tau, U\right)$ is upper hemicontinuous. Given the assumption of $2c''\left(e\right) + ec'''\left(e\right)$, the same argument for proposition 7 shows that $\theta\left(\tau, U\right)$ is single valued. Hence, $\theta\left(\tau, U\right)$ is continuous. It means that $B\left(U\right)$ is continuous. Suppose $U = \frac{y_H - \varepsilon}{1-\beta}$, where $\varepsilon > 0$. Then $S_H = \frac{\varepsilon}{1-\beta(1-\lambda)}$. Because $y_H > y_L$, we can choose ε so that $\frac{y_L}{1-\beta} < U = \frac{y_H - \varepsilon}{1-\beta}$. Then $S_0\left(\theta\left(\tau, \frac{y_H-\varepsilon}{1-\beta}\right), \frac{y_H-\varepsilon}{1-\beta}\right) = e^{\tau}\frac{\varepsilon}{1-\beta(1-\lambda)} - c\left(e^{\tau}\right)$. As $e^{\tau} \in (0,1)$ for any $\varepsilon > 0$, we can choose ε so that $S_0\left(\theta\left(\tau, \frac{y_H-\varepsilon}{1-\beta}\right), \frac{y_H-\varepsilon}{1-\beta}\right)$ are close to 0. That is, we can find $\varepsilon > 0$ so that $B\left(\frac{y_H-\varepsilon}{1-\beta}\right) > 0$. Suppose $U = 0$. Similar to the argument with the proof of theorem 9, it is clear that the contract-posting problem can be $\frac{\beta p(\theta(\tau,U)) R_0(\mathbf{R})}{1-\beta} > 0$ for any U. Therefore, $B\left(0\right) < 0$. This proves the existence.

Note that

$$B'\left(U\right) = 1 - \left[\frac{\partial \hat{U}\left(\theta, U\right)}{\partial U} + \frac{\partial \hat{U}\left(\theta, U\right)}{\partial \theta}\frac{\partial \theta\left(\tau, U\right)}{\partial U}\right].$$

Now $\frac{\partial \hat{U}(\theta,U)}{\partial U} = \frac{\beta p(\theta^\tau)\frac{\partial S_0(\theta,U)}{\partial U}}{1-\beta}$ where

$$\frac{\partial S_0(\theta,U)}{\partial U} = S_H'(U) + (1-e^\tau)\hat{S}_L'(U) + \left[\left(S_H(U) - \hat{S}_L(U)\right) - c'(e)\right]\frac{\partial e(\theta,U)}{\partial U},$$

and we can derive from the zero-profit–IC constraint that

$$\frac{\partial e(\theta,U)}{\partial U} = -\frac{I\left[e < e^{best}\right]\left[eS_H'(U) + (1-e)\hat{S}_L'(U)\right]}{\left[S_H(U) - \hat{S}_L(U) - c'(e) - ec''(e)\right]}.$$

Hence, $\frac{\partial e(\theta,U)}{\partial U} \leq 0$ with a strict inequality if $e < e^{best}$, and $\frac{\partial \hat{U}(\theta,U)}{\partial U} < 0$. Moreover, because the first-order condition with respect to θ implies $\beta p'(\theta^\tau) S_0(\theta^\tau, U^\tau) - k + \beta p(\theta^\tau)\frac{\partial S_0(\theta^\tau,U^\tau)}{\partial \theta} = \tau$, it can be shown that $\frac{\partial \hat{U}(\theta,U)}{\partial \theta} = \frac{\tau}{1-\beta}$. Hence, we can find $\hat{\tau} \in (0,\infty]$ such that, for any $\tau < \hat{\tau}$, $B'(U) > 0$. This proves the uniqueness.

Knowing that $n_{H,t}^E = e_t n_{0,t}^B + n_{H,t}^B$ and $n_{L,t}^E = x_t(1-e_t)n_{0,t}^B + n_{L,t}^B$, it is shown that

$$\begin{aligned}\frac{dY\left(\tau : \mathbf{N}_t^B\right)}{d\tau} &= \left[1 + S_H'(U^\tau)n_{H,t}^E + S_L'(U^\tau)n_{L,t}^E\right]\frac{dU^\tau}{d\tau} \\ &+ n_{0,t}^B\left[S_H^S(U^\tau) - \hat{S}_L^S(U^\tau) - c'(e(\theta,U^\tau))\right]\frac{\partial e(\theta,U^\tau)}{\partial \theta}\frac{d\theta(\tau,U^\tau)}{d\tau} \\ &+ n_{0,t}^B\left[(S_H(U^\tau) - S_L(U^\tau) - c'(e(\theta,U^\tau)))\frac{\partial e(\theta,U^\tau)}{\partial U^\tau}\right]\frac{dU^\tau}{d\tau}.\end{aligned}$$

Moreover, because it is easy to show that $1 + S_H'(U^\tau)n_{H,t}^E + S_L'(U^\tau)n_{L,t}^E = \frac{\beta\lambda + (1-\beta)u_t^E}{1-\beta(1-\lambda)}$ and it is derived from $U^\tau = \hat{U}(\theta(\tau,U^\tau),U^\tau)$ that $\frac{dU^\tau}{d\tau} = \frac{\frac{\partial \hat{U}(\theta^\tau,U^\tau)}{\partial \theta^\tau}\frac{\partial \theta(\tau,U^\tau)}{\partial \tau}}{B'(U)}$, we can derive equation (24) and $\frac{\frac{d\theta(\tau,U^\tau)}{d\tau}}{\frac{\partial \theta(\tau,U^\tau)}{\partial \tau}} = 1 + \frac{\frac{\partial \theta(\tau,U^\tau)}{\partial U^\tau}\frac{\partial \hat{U}(\theta^\tau,U^\tau)}{\partial \theta^\tau}}{B'(U)} > 0$. Finally, it is derived from the first-order condition with respect to θ, $\beta p'(\theta^\tau) S_0(\theta^\tau,U^\tau) - k + \beta p(\theta^\tau)\frac{\partial S_0(\theta^\tau,U^\tau)}{\partial \theta} = \tau$, that

$$\frac{\partial \theta(\tau,U)}{\partial \tau} = \frac{1}{\beta\left[p''(\theta)S_0(\theta,U) + 2p'(\theta)\frac{\partial S_0(\theta,U)}{\partial \theta} + p(\theta)\frac{\partial^2 S_0(\theta,U)}{\partial \theta^2}\right]},$$

where

$$\begin{aligned}\frac{\partial S_0(\theta,U)}{\partial \theta} &= \left[S_H(U) - \hat{S}_L(U) - c'(e(\theta,U))\right]\frac{\partial e(\theta,U)}{\partial \theta}, \\ \frac{\partial^2 S_0(\theta^\tau,U^\tau)}{\partial \theta^2} &= \left[S_H(U) - \hat{S}_L(U) - c'(e(\theta,U))\right]\frac{\partial^2 e(\theta,U)}{\partial \theta^2} - c''(e(\theta,U))\left[\frac{\partial e(\theta,U)}{\partial \theta}\right]^2.\end{aligned}$$

Because it is shown from the zero-profit–IC constraint that

$$\begin{aligned}\frac{\partial e(\theta,U)}{\partial\theta} &= -\frac{I\left[e<e^{best}\right]q'(\theta)\left[e\left(S_H(U)-\hat{S}_L(U)-c'(e)\right)+\hat{S}_L(U)-\hat{R}_L\right]}{q(\theta)\left[S_H(U)-\hat{S}_L(U)-c'(e)-ec''(e)\right]},\\ \frac{\partial^2 e(\theta,U)}{\partial\theta^2} &= M\frac{\partial e(\theta,U)}{\partial\theta},\ for\ some\ M>0,\end{aligned}$$

$\frac{\partial e(\theta,U)}{\partial\theta}\le 0$ and $\frac{\partial^2 e(\theta,U)}{\partial\theta^2}\le 0$ with a strict inequality if $e<e^{best}$. Therefore, $\frac{\partial S_0(\theta,U)}{\partial\theta}\le 0$, $\frac{\partial^2 S_0(\theta^\tau,U^\tau)}{\partial\theta^2}\le 0$, and $\frac{\partial\theta(\tau,U)}{\partial\tau}<0$. **Q.E.D.**

Proof of Proposition 16**:** Let $\left(\theta^*,e^*,\hat{R}_L^*,U^*\right)$ be the solution to the competitive search equilibrium and $\left(\theta\left(\alpha,U^B\right),e^B\left(\alpha,U^B\right),U^B\right)$ be the solution to the search model with wage bargaining. Suppose that $y_L<y_L^{cc}$. Then $\hat{R}_L^*=\hat{S}_L(U^*)=0$. Define α^* such that $\alpha^*=\frac{c'(e^*)}{S_H(U^*)}$. Then because $c'\left(e^B(\alpha^*,U)\right)=\alpha^*\left(S_H(U)-\hat{S}_L(U)\right)$, $\frac{\partial e^B(\alpha^*,U)}{\partial U}\le 0$, and $\frac{\partial e^B(\alpha^*,U)}{\partial\alpha}>0$. We first show that $U^B=U^*$ when $\alpha=\alpha^*$ and later show that for a small $\varepsilon>0$, the search model with wage bargaining with $\alpha^*+\varepsilon$ attains higher welfare than that with α^*. Define $q^B(\alpha,U)=q\left(\theta^B(\alpha,U)\right)$ and $\tilde{p}\left(q^B(\alpha,U)\right)=p\left(\theta^B(\alpha,U)\right)$. Define also

$$\begin{aligned}D^B(U) &= U-\frac{\tilde{p}\left(q^B(\alpha^*,U)\right)\beta\left[e^B(\alpha^*,U)c'\left(e^B(\alpha^*,U)\right)-c\left(e^B(\alpha^*,U)\right)+\alpha^*\hat{S}_L(U)\right]}{1-\beta},\\ q^B(\alpha^*,U) &= \frac{k}{\beta\left[e^B(\alpha^*,U)\left(S_H(U)-\hat{S}_L(U)-c'\left(e^B(\alpha^*,U)\right)\right)+(1-\alpha^*)\hat{S}_L(U)\right]}.\end{aligned}$$

Then the search model with wage bargaining when $\alpha=\alpha^*$ is shown to be the solution to $D^B\left(U^B\right)=0$. We can easily check that $D^B(0)<0$, $D^B\left(\frac{y_H-\varepsilon}{1-\beta}\right)>0$ for small ε, and $D^{B\prime}(U)<0$. Hence, there is a unique U^B that satisfies $D^B\left(U^B\right)=0$. Consider the case $U=U^*$. Note that $\hat{R}_L^*=\hat{S}_L(U^*)=0$. Because $c'\left(e^B(\alpha^*,U^*)\right)=\alpha^*S_H(U^*)=c'(e^*)$, $e^B(\alpha^*,U^*)=e^*$. Moreover, because $q\left(\theta^B(\alpha^*,U^*)\right)=\frac{k}{\beta e^*[S_H(U^*)-c'(e^*)]}=q(\theta^*)$, $\theta^B(\alpha^*,U^*)=\theta^*$. Finally, because

$$\begin{aligned}&\frac{\tilde{p}\left(q^B(\alpha^*,U^*)\right)\beta\left[e^B(\alpha^*,U^*)c'\left(e^B(\alpha^*,U^*)\right)-c\left(e^B(\alpha^*,U^*)\right)+\alpha^*\hat{S}_L(U^*)\right]}{1-\beta}\\ =\ &\frac{\tilde{p}(q(\theta^*))\beta\{e^*c'(e^*)-c(e^*)\}}{1-\beta}=U^*,\end{aligned}$$

$D^B(U^*) = 0$. As we have shown that U^B is unique, $U^B = U^*$, $e^B\left(\alpha^*, U^B\right) = e^*$, $\theta^B\left(\alpha^*, U^B\right) = \theta^*$ and, therefore, $S_0^B\left(\alpha^*, U^B\right) = S_0(\theta^*, U^*)$. This means that when $\alpha = \alpha^*$, the solutions to the search model with wage bargaining are the same as the competitive search equilibrium, and that the search model with wage bargaining under $\alpha = \alpha^*$ attains the same level of welfare as the competitive search equilibrium. Consider $\frac{dY^B(\hat{N}_t)}{d\alpha}|_{\alpha=\alpha^*} = \frac{\partial Y^B(N_t)}{\partial U^B}\frac{\partial U^B}{\partial \alpha}|_{\alpha=\alpha^*} + \frac{\partial Y^B(N_t)}{\partial \alpha}|_{\alpha=\alpha^*}$. We can easily check that

$$\begin{aligned}\frac{\partial Y^B\left(\hat{N}_t\right)}{\partial \alpha}|_{\alpha=\alpha^*} &= \frac{\partial S_0^B\left(\alpha, U^B\right)}{\partial \alpha}|_{\alpha=\alpha^*} n_{0.t}^B = \left[S_H(U^*) - c'(e^*)\right]\frac{\partial e^B(\alpha^*, U^*)}{\partial \alpha} n_{0.t}^B > 0, \\ \frac{\partial U^B}{\partial \alpha}|_{\alpha=\alpha^*} &= \frac{\partial U^B}{\partial e}|_{e=e^*}\frac{\partial e^B(\alpha^*, U^*)}{\partial \alpha}.\end{aligned}$$

Because we can show that $\frac{\partial U^B}{\partial e}|_{e=e^*} = 0$,

$$\frac{dY^B\left(\hat{N}_t\right)}{d\alpha}|_{\alpha=\alpha^*} = \frac{\partial Y^B(N_t)}{\partial \alpha}|_{\alpha=\alpha^*} = \left[S_H(U^*) - c'(e^*)\right]\frac{\partial e^B(\alpha^*, U^*)}{\partial \alpha} n_{0.t}^B > 0.$$

Therefore, for a small $\varepsilon > 0$, the search model with wage bargaining under $\alpha^* + \varepsilon$ attains higher welfare than that with α^*. **Q.E.D.**

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